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A COEFFICIENT OF LINEAR CORRELATION BASED ON THE METHOD OF LEAST SQUARES AND THE LINE OF BEST FIT.

By J. B. Coleman

Given N points in a plane, corresponding to N pairs of values for two variables, X and Y, we find the line of best fit and the line of *worst* fit, by the method of least squares*. Then we derive a cofficient of correlation based on the sum of the squares of the distances of the points from these two lines.

The line of best fit is in the line such that the sum of the squares of the distances of the points from it is a minimum. The line of *worst* fit is the one from which the sum of the squares of the distances of the points is a maximum. We shall refer to them, respectively, as the *minimum* and *maximum* lines.

For convenience we take the origin at the centre of gravity of the points, letting x and y denote deviations of X and Y, respectively, from their arithmetic means.

1. The two lines pass thru the arithmetic means of the X's, and of the Y's.

$y = mx + b$ may represent any line of the plane. The distance, d_i, of a point, (x_i, y_i), from the line is $\frac{y_i - mx_i - b}{\sqrt{1+m^2}}$. The sum of the squares of the distances of the N points from the line will be

*For a general discussion of this method of fitting when q variables are involved, see, Pearson, Karl, "On Lines and Planes of Closest Fit to Systems of Points in Space", Phil. Mag., 6th series, vol. ii, 1901, P. 559.

$$(1) \qquad \Sigma d^2 = \frac{\Sigma y^2 + m^2 \Sigma x^2 + Nb^2 - 2m\Sigma xy - 2b\Sigma y + 2mb\Sigma x}{1+m^2}$$

Using $\Sigma x = \Sigma y = 0$, in the condition for maximum or minimum values in (1), the condition reduces to $b = 0$, and the theorem follows.

2. To find the slopes of the two lines.
Equation (1) now becomes

$$(2) \qquad \Sigma d^2 = \frac{\Sigma y^2 - 2m\Sigma xy + m^2 \Sigma x^2}{1+m^2}.$$

The condition under which (2) will have a maximum or minimum values, is that

$$m^2 \Sigma xy + m(\Sigma x^2 - \Sigma y^2) - \Sigma xy = 0.$$

This condition is satisfied by two values of m, namely;

$$(3) \qquad m_1 = \frac{\Sigma y^2 - \Sigma x^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}{2\Sigma xy},$$

$$(4) \qquad m_2 = \frac{\Sigma y^2 - \Sigma x^2 - \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}{2\Sigma xy}.$$

It is found by considering the second derivative that (4) is the condition under which Σd^2 will have a maximum value, and (3) is the condition for a minimum value.

The equation of the *minimum* line is $y = m_1 x$, and that of the *maximum* line is $y = m_2 x$. The value of m_1 and m_2 are those given in (3) and (4).

3. The *maximum* and *minimum* lines are perpendicular to each other.

That $m_1 = -1/m_2$ is easily shown from (3) and (4).

Further m_1 has the same sign as Σxy, and m_2, the opposite sign, since their numerators are, respectively, positive and negative.

4. The *minimum* line lies between the two lines of regression, or coincides with them.

If $\Sigma xy > 0$,

$$m_1 \leqq \frac{\Sigma y^2 - \Sigma x^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4\Sigma x^2 \Sigma y^2}}{2\Sigma xy}$$

since $\Sigma x^2 \Sigma y^2 \geqq (\Sigma xy)^2$.

Hence $m_1 \leqq \Sigma y^2 / \Sigma xy = m_{xy}$, the slope of the line of regression of x on y.

Rationalizing the numerator of (3), and noting that $\Sigma x^2 \Sigma y^2 \geqq (\Sigma xy)^2$, we obtain

$$m_1 \geqq \frac{2\Sigma xy}{\Sigma x^2 - \Sigma y^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4\Sigma x^2 \Sigma y^2}} = \frac{\Sigma xy}{\Sigma x^2} = m_{yx},$$

the slope of the line of regression of y on x.

In the same way it may be shown that if $\Sigma xy < 0$, then $m_{xy} \leqq m_1 \leqq m_{yx}$.

The condition that m_1 be equal to the slope of one line of regression is the same that it be equal to the other, so that the *minimum* line coincides with both lines of regression, or else lies between the two.

5. To find the sum of the squares of the distances of the points from the *minimum* line; also from the *maximum* line.

Let d be the distance of a point from the line $y = m_1 x$

$$\Sigma d^2 = \sum \left(\frac{y - m_1 x}{\sqrt{1 + m^2}} \right)^2 = \frac{\Sigma y^2 - 2 m_1 \Sigma xy + m_1^2 \Sigma x^2}{1 + m_1^2} .$$

Substituting for m from (3) and reducing, we obtain

(5)
$$\Sigma d^2 = \left[\Sigma x^2 + \Sigma y^2 - \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2} \right] / 2 .$$

Similarly, if D represents the distance of a point from the line $y = m_2 x$, we obtain

(6) $$\Sigma D^2 = \left[\Sigma x^2 + \Sigma y^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2} \right] / 2 .$$

6. To find a coefficient of linear correlation.

Let $q = \sqrt{\Sigma d^2 / \Sigma D^2}$. Substituting from (5) and (6), and reducing, we obtain

(7a) $$q = \frac{\Sigma x^2 + \Sigma y^2 - \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}{2\sqrt{\Sigma x^2 \Sigma y^2 - (\Sigma xy)^2}} ,$$ or

(7b) $$q = \frac{2\sqrt{\Sigma x^2 \Sigma y^2 - (\Sigma xy)^2}}{\Sigma x^2 + \Sigma y^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}$$

q represents the ratio of the root-mean-squares of the distances of the point from the *minimum* and *maximum* lines. This ratio

is a measure of the closeness of fit of the points to a line, and should furnish a measure of linear correlation. The value of q may vary from 0 to 1. $q = 0$ indicates that the points are all on a straight line, hence that the correlation is perfect. It is of interest to note that when $q = 0$, (7b) gives $\Sigma xy/N\sigma_x\sigma_y$ $[= r] = \pm 1$. When q is 1 the mean squares of the distances of the points from two lines at right angles is the same, and linear correlation is lacking. Hence $1-q$ would furnish a coefficient conforming to the customs that it have the value 1 for perfect correlation, and 0 for lack of correlation.

Values of q found from (7a) or (7b) would involve the units in which X and Y are given. Hence these forms would be objectionable, in that q could be made to assume different values for the same data, by changing the units in which x and y are expressed. However, this objection may be removed by taking σ_x and σ_y as units in which to express X and Y. (7b) then reduces to

$$q = \frac{\sqrt{N^2 - (\Sigma xy)^2/\sigma_x^2\sigma_y^2}}{N + |\Sigma xy/\sigma_x\sigma_y|}.$$

The coefficient $1-q$ may now be expressed as

$$(8) \qquad r_c = 1 - q = 1 - \sqrt{\frac{N - |\Sigma xy/\sigma_x\sigma_y|}{N + |\Sigma xy/\sigma_x\sigma_y|}}.$$

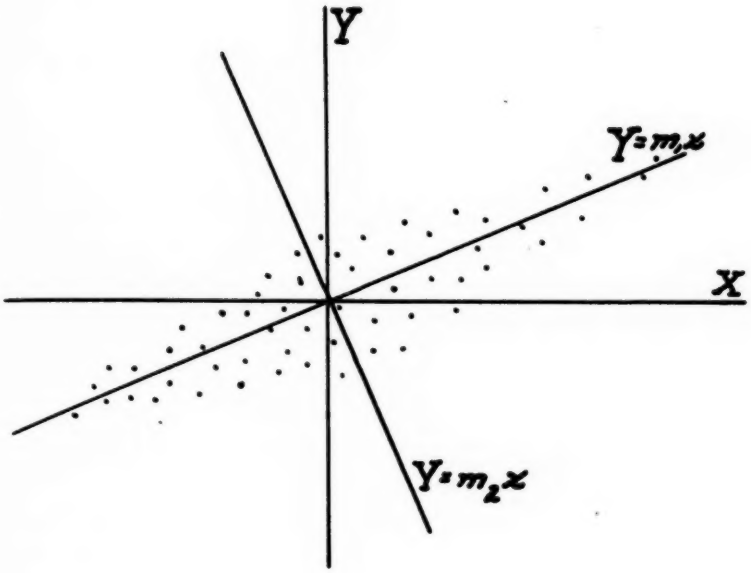
The sign of the coefficient should agree with that of the slope of the line to which the points are fitted. Hence, when the value for $1-q$ has been found it should be given the same sign as the slope of the *minimum* line. But the slope of the *minimum* line is determined by that of Σxy, so that the sign given to $1-q$ should be that of Σxy.

The coefficient $1-q$ may be expressed immediately in terms of the Pearson coefficient, r, which is equal to $\Sigma xy/N\sigma_x\sigma_y$. Making this substitution in (8) we have

$$r_c = 1-q = 1-\sqrt{\frac{1-|r|}{1+|r|}}\ .$$

In the table are shown values of $1-q$ corresponding to some given values of r. The values for $1-q^2$ have also been listed corresponding to the same set of values for r. The maximum difference occurs between $1-q$ and r when $r=.839$ and $1-q=.704$, a difference of .135 by which $1-q$ is smaller.

r	$r_c=1-q$	$1-q^2$
1	1	1
.99	.929	.995
.95	.840	.974
.9	.771	.947
.839	.704	.912
.8	.667	.889
.7	.580	.824
.6	.500	.750
.5	.423	.667
.4	.345	.571
.3	.266	.462
.2	.183	.333
.1	.095	.182
0	0	0



J. B. Coleman

A STUDY OF THE DISTRIBUTION OF MEANS ESTIMATED FROM SMALL SAMPLES BY THE METHOD OF MAXIMUM LIKELIHOOD FOR PEARSON'S TYPE II CURVE

By JOHN L. CARLSON

The object of this paper is to study the distribution of estimates of the parameter of location for Pearson's Type II Curve, estimated by the method of maximum likelihood from small samples.

R. A. Fisher has assumed,[1] and Professor Hotelling has proved[2] that in large categories of cases the distribution of an optimum statistic approaches normality as the sample size increases. This normality has been assumed to hold for optimum statistics in general whether calculated from large samples or small ones, and it has also been assumed that optimum statistics have minimum variance and always give better fits than do statistics calculated by the method of moments. That this is the case whenever the sample is large and the distribution of optimum statistics normal is made plausible by the reasoning of R. A. Fisher.[3] In case the sample is small, however, there may be reason to doubt that the normality of distribution of optimum statistics holds, and that the other conclusions hold. It is with this phase of the subject that we shall be concerned in what follows.

Before entering into the topic under discussion it will be convenient to review some of the more elementary facts regarding the curve with which we are to be concerned. We shall take first the general equation for the curve in the form,

$$(1) \qquad y = y_0\left(1 - \frac{(x-m)^2}{a^2}\right)^p$$

[1]On the Mathematical Foundations of Theoretical Statistics, R. A. Fisher, Phil. Trans. Series A, Vol. 222, 1922. Pp. 309-368.

[2]The Consistency and Ultimate Distribution of Optimum Statistics. Harold Hotelling, Trans. Amer. Math. Soc. Vol. 32, No. 4. Pp. 847-859.

[3]Ibid, Pp. 328-368.

and determine the effect of variation of the constants.

When $p=0$ the equation reduces to the straight line $y=y_0$

When $p=+1$ the equation is that of a parabola with $y=y_0$ at the point $x=m$ and the x intercepts at the points $x=\pm a$. It of course meets the x axis at an angle.

When $p=+2$ the equation $y=0$ is of the fourth degree in x with double roots at the points $x=\pm a$.

In general when $p=n$ the equation $y=0$ is of degree $2n$ in x and has two sets of n-fold multiple roots $x=\pm a$.

Since our curve is to be a probability curve it will of necessity have unit area, and this fact makes it possible for us to evaluate y_0 in terms of the parameters a and p. In order to do this we shall perform the integration below.

$$\text{Area} = 1 = \int_{x-a}^{x+a} y\,dx = 2y_0 \int_0^a \left(1-\frac{x^2}{a^2}\right)^p dx$$

whence

$$1 = 2ay_0 \int_0^{\frac{\pi}{2}} \cos^{2p+1}\theta\,d\theta$$

but now since

$$\int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \tag{2}$$

we have $$1 = 2ay_0\frac{\sqrt{\pi}}{2}\,\frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}$$

and so

$$y_0 = \frac{\Gamma\left(p+\frac{3}{2}\right)}{a\sqrt{\pi}\,\Gamma(p+1)}$$

Therefore

$$(3) \qquad y = \frac{\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)}\left[1-\frac{(x-m)^2}{a^2}\right]^p$$

Formula (2) we shall find to be of value a number of times and formula (3) is the form in which equation (1) will be used throughout the remainder of the paper.

It will be worth while now to consider the likelihood function L together with its first and second partial derivatives with respect to m. (We shall hereafter refer to m as the parameter of location, a as the parameter of scaling, and p as the parameter of shape.) We are to use m to denote the estimate of m obtained by the method of maximum likelihood, in accordance with the convention introduced by Fisher,[1] and it will be with this parameter that we shall concern ourselves in this investigation.

We have from (3) on the preceding page

$$(4) \qquad L = n \log \frac{\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)} + p\sum_{i=1}^{n} \log\left[1-\frac{(x_i-m)^2}{a^2}\right]$$

and so

$$(5) \qquad \frac{\partial L}{\partial m} = 2p\sum_{i=1}^{n} \frac{x_i-m}{a^2-(x_i-m)^2}$$

and

$$(6) \qquad \frac{\partial^2 L}{\partial m^2} = -2p\sum_{i=1}^{n} \frac{a^2+(x_i-m)^2}{[a^2-(x_i-m)^2]^2}$$

[1] Ibid Pp. 309-368.

At this point we shall stop to consider the effect of variation of the parameters a and ρ upon our estimate of $\hat{m}$. Let us first consider ρ. Since the method of maximum likelihood is here merely the method of the differential calculus it follows from a consideration of equation (5) that our estimate of $\hat{m}$ will be independent of ρ, for any particular sample. Such is not the case when we consider a, however, for any change in a allows a change in the variance of $\hat{m}$ for the particular sample.

We shall find it advantageous to cover as much of the theoretical work as possible before embarking upon our experimental check and its great amount of numeral calculations, for it is only by means of a check between theory and experiment that we are able in the present state of knowledge, to judge of the applicability of the method of maximum likelihood to small samples. Of course it will be necessary to consider the distribution of our estimates of $\hat{m}$, in order to make this statistic of practical use, and it is desirable to know the theoretical variances of $\bar{x}$ the arithmetic mean, $\hat{m}$, and the experimentally obtained variance of the distribution of our estimates of $\hat{m}$. The first of these we can obtain from theory, the second from an approximation valid only in the limit, and the last by means of calculation based on actual sampling.

We shall be concerned first with the theoretical variance of $\bar{x}$. It is well known that the variance of $\bar{x}$ is equal to the variance of the distribution divided by n. We must first, therefore, find the variance of the distribution.

From (3) page 88 it follows that

$$\sigma^2 = \frac{2\Gamma(\rho+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(\rho+1)} \int_0^a x^2 \left(1-\frac{x^2}{a^2}\right)^{\rho} dx$$

$$\sigma^2 = \frac{2\Gamma(\rho+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(\rho+1)} \int_0^{\frac{\pi}{2}} \left[\cos^{2\rho+1}\theta - \cos^{2\rho+3}\theta\right] d\theta$$

remembering now (2) page 87

$$\sigma^2 = \frac{a^2}{2p+3}$$

whence it follows that

$$(7) \qquad \sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{a^2}{n(2p+3)}.$$

We shall now calculate the limiting form of the variance of $\hat{m}$. Fisher has proved[1] that if the distribution of optimum statistics is normal the variance of an optimum statistic is equal to the negative reciprocal of the mathematical expectation of the second partial derivative of the logarithm of the likelihood with respect to the parameter in question.

We may write, therefore

$$-\frac{1}{\sigma_{\hat{m}}^2} = \frac{-4np\,\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)} \int_0^a \frac{(a^2+x^2)}{(a^2-x^2)} \left[1-\frac{x^2}{a^2}\right]^p dx$$

whence

$$-\frac{1}{\sigma_{\hat{m}}^2} = \frac{-4np\,\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)} \int_0^{\frac{\pi}{2}} (2\cos^{2p-3}\theta - \cos^{2p-1}\theta)\,d\theta$$

and so again referring to (2) page 87 we have

$$-\frac{1}{\sigma_{\hat{m}}^2} = -\frac{2np\,(p+\frac{1}{2})}{a^2(p-1)}$$

hence we have

$$(8) \qquad \sigma_{\hat{m}}^2 = \frac{a^2(p-1)}{np(2p+1)}$$

The efficiency of the mean is then

$$(9) \qquad E = \frac{(p-1)(2p+3)}{p(2p+1)} = \frac{\sigma_{\hat{m}}^2}{\sigma_{\bar{x}}^2}$$

The next problem with which we must concern ourselves is that of experimental verification of the assumptions under discus-

[1] Ibid Pp. 327-328.

sion. The problem is briefly that of choosing a number of small samples from a population which obeys our law of frequency, estimating m for these samples, and then calculating the variance for the distribution. Also we shall draw an histogram of the distribution and observe the general type of the distribution, so nearly as that is possible from our samples.

The problem of choosing our samples is not the least of our difficulties, for we can not take all types of samples. They must be of a very special nature: they must be from a population of the Type II. In order to accomplish this it will be necessary to have a table of areas corresponding to given values of x for the Type II Curve. There is no such table available to the knowledge of the writer, and it is therefore necessary to construct the table before we can procede with the choosing of the samples. After the table has been built, we can with the aid of Tippett's Tables,[1] choose our samples with ease. The manner of choosing is as follows. Take the numbers from Tippett's Tables as areas under the Type II Curve and look up in the table of areas the values of x corresponding to the smallest area containing the area found from the Random Numbers. This will give the value of x to be taken. Since we will take four digits let the fifth digit determine the sign. If it is odd take the sign $-$, if it is even take the sign $+$.

There are two ways in which a table of this nature can be prepared, and the method employed must in any case be determined by the degree of accuracy attainable and the amount of labor involved. One of these methods is that of the calculus of finite differences, determining the zero order differences by means of algebra, and from these by the process of addition building up the table. This method is best used when a dependable listing adding machine is at hand. The other method is that of direct integration, and it is found that with the aid of two calculating machines this is by far the quicker. It was this method that was applied in the building of the table on page 92 and 93.

[1]Tracts For Computers, No. XV.

Table of Areas Under Pearson's Type II Curve, Correct to 9 Places of Decimals. The Areas are included between ordinates located $\pm x$ units from the parameter of location.

Constants. $y_o = {}^{15}/_{16}a$, $m=0$, $p=2$, and $a=1$

x	Area	x	Area
0.00	0.000 000 000	30	529 661 250
01	018 748 750	31	545 084 843
02	037 490 001	32	560 298 291
03	056 212 259	33	575 295 077
04	074 920 038	34	590 073 828
05	093 593 867	35	604 625 820
06	112 230 292	36	618 947 482
07	130 821 880	37	633 034 148
08	149 361 229	38	646 881 319
09	167 840 964	39	660 484 657
10	186 253 750	40	673 840 000
11	204 592 289	41	686 943 358
12	222 849 331	42	699 790 921
13	241 017 673	43	712 379 067
14	259 090 168	44	724 704 358
15	277 059 727	45	736 763 555
16	294 919 322	46	748 553 612
17	312 661 995	47	760 071 688
18	330 280 859	48	773 151 488
19	347 769 104	49	782 281 572
20	365 120 040	50	792 968 750
21	382 326 904	51	803 374 696
22	399 383 261	52	813 497 651
23	416 282 613	53	823 336 081
24	433 018 598	54	832 888 688
25	449 584 961	55	842 154 414
26	465 975 552	56	851 132 442
27	482 184 334	57	860 009 702
28	498 205 389	58	868 223 379
29	514 032 918	59	876 335 911
30	529 661 250	60	884 160 000

61	892 546 290	81	985 203 165
62	898 944 981	82	987 317 441
63	905 907 620	83	989 230 274
64	912 585 318	84	990 949 478
65	919 120 273	85	992 483 242
66	925 092 472	86	993 840 132
67	930 925 941	87	995 029 095
68	936 482 509	88	996 059 469
69	941 764 926	89	996 940 979
70	946 776 250	90	997 683 750
71	951 519 851	91	998 298 304
72	955 999 411	92	998 795 571
73	960 294 310	93	999 186 888
74	964 182 748	94	999 484 008
75	967 895 508	95	999 699 102
76	971 362 202	96	999 844 762
77	974 588 156	97	999 934 010
78	977 579 039	98	999 980 299
79	980 340 865	99	999 997 519
80	982 880 000	100	1.000 000 000

The table on the preceding page was built by direct integration of (3) page 88, with $p=2$ and $a=1$ These values for the parameters were chosen so as to save as much labor in calculation as possible, and at the same time maintain the desired shape of the curve. There has of course been no less of generality in setting $a=1$ but we have limited ourselves quite definitely in using the value $p=2$. The accuracy of the table to 9 places of decimals has been assured by calculating all values to 13 places and then determining the maximum error over the whole range which was 625 in the 13 place due to the use of the decimal equivalent of $^2/_3$ in the third degree term.

Our table of areas being complete and the problem of sampling thus solved, we must next consider the task of estimating $\hat{m}$. Since we have already taken $a=1$ and $p=2$ in building our tables we shall continue to use these values throughout the work.

The next question to be settled before beginning our work is the size of the samples with which we are to deal. The case $n = 1$ is of course of no interest for the best estimate of a single observation is the observation itself. The case $n = 2$ is likewise of no interest for in this case the arithmetic mean coincides with the solution by the method of maximum likelihood. Therefore it is the case $n = 3$ with which we shall be concerned. Our results are not then trivial, and at the same time we have the case which is the easiest to deal with, since the number of numerical calculations for each sample is reduced to the minimum.

Before going ahead with any attempt at solving an equation such as (5) page 88, it is well to have in mind a picture of what we are actually trying to accomplish. With such a picture in mind we are better able to realize the difficulties of the situation and so are better able to cope with them. To this end we have included a graph of the problem involved which will make clear at a glance just what must be done to find the true value of m.

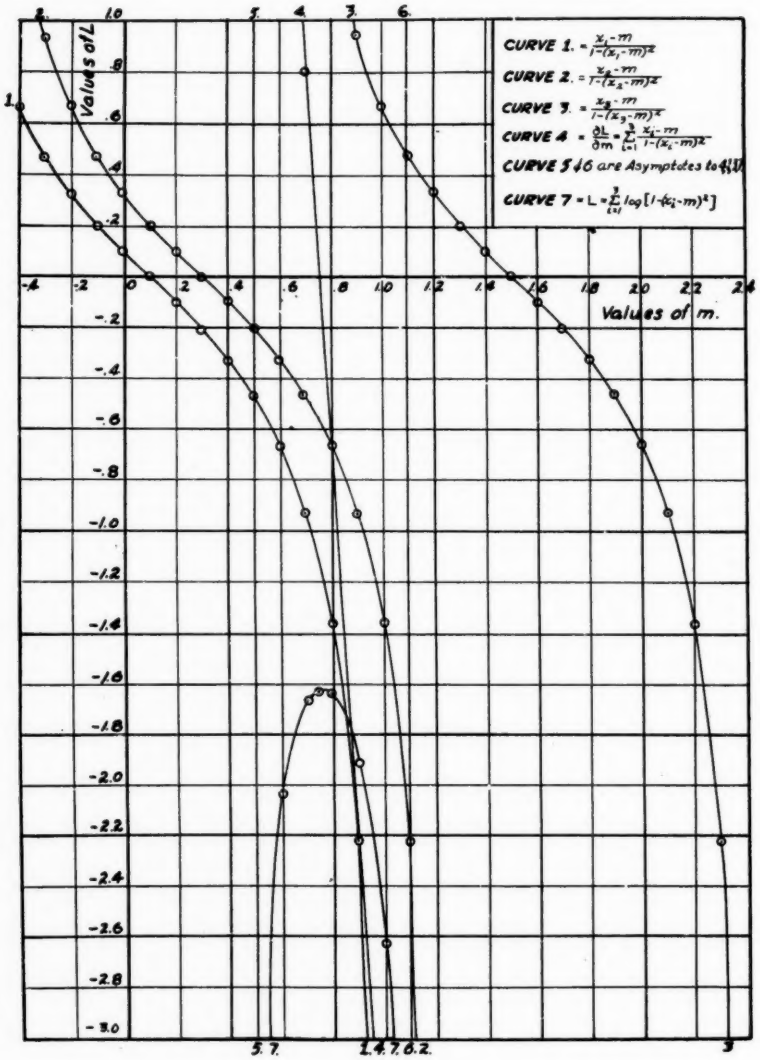
We have drawn, page 96 , from plotted points the curves representing L , $\frac{\partial L}{\partial m}$ and the three terms of $\frac{\partial L}{\partial m}$. Also we have drawn in the asymptotes to the curve which are of significance. If we are able to find the point at which $\frac{\partial L}{\partial m} = 0$ we have the solution to our problem. This involves solving a fifth degree equation. We shall use Newton's method of successive approximation. Fisher states[1] that in some cases at least, we may start with an inefficient statistic and by a single approximation obtain an efficient one. Whether or not this is the case for small samples we shall see when our calculations have been analyzed.

It will be seen upon examining the graph that as is allowed to vary each of the therms of $\frac{\partial L}{\partial m}$ varies from $-\infty$ to $+\infty$,

[1]Theory of Statistical Estimation, R. A. Fisher, Proc. Cam. Phil. Soc. Vol. XXII part 5. Pp. 708-709.

and so their sum also varies between these limits. It will be seen that the asymptote corresponding to the largest allowable value of m can be found by adding the value of a to the smallest observation, and that the asymptote corresponding to the smallest allowable value of m may be found by subtracting the value of a from the greatest observation. It is thus evident that as dispersion in the sample increases the variance of m must surely decrease. That this fact is of fundamental importance in choosing our first estimate of m will be seen from consideration of the following case, since it is well known that in the case of a curve with a real finite pole Newton's method may lead us to erroneous results. Let us consider the sample consisting of the three observations $x_1 = -.99$, $x_2 = +.99$, $x_3 = +.99$, for which the arithmetic mean is $\bar{x} = +.33$. If now we take as our first approximation $m_1 = \bar{x}$ we will immediately lead ourselves to an erroneous value of m_2, our estimate of $\hat{m}$. That this is the case will be easily seen by means of the check given on page 94 for using $a = 1$ we locate the asymptotes at $x = \pm .01$. The true value of m located between these asymptotes, is not therefore even to be approached should we take $m_1 = \bar{x}$. This is an extreme case and fortunately not to be anticipated very often, or the arithmetic mean would be deprived of any value whatsoever. We should always be sure that the difference between any observation and the value of m that we are using is not greater than the value of a when dealing with the Type II Curve. This holds no matter what our manner of attack may be.

We shall now be concerned with the calculations for our 100 samples of 3. All of the data are tabulated in such a way as to be self-explanatory, and so we shall not bother to give sample calculations. The next ten pages cover these calculations. The discussion is continued on page 104.


Values of L
Values of m.
CURVE 1. = $\frac{x_1 - m}{1-(x_1-m)^2}$
CURVE 2. = $\frac{x_2 - m}{1-(x_2-m)^2}$
CURVE 3. = $\frac{x_3 - m}{1-(x_3-m)^2}$
CURVE 4 = $\frac{\partial L}{\partial m} = \sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$
CURVE 5 & 6 are Asymptotes to 4 & 7
CURVE 7 = $L = \sum_{i=1}^{3} \log[1-(x_i-m)^2]$
1.0
.8
.6
.4
.2
-.2
-.4
-.6
-.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
-2.4
-2.6
-2.8
-3.0
-.4
-.2
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
1
2
3
4
5
6
5.7.
2.4.7.6.2.
3

Sample No	i	x_i	$m_1 = \bar{x} = \frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1-(x_i-m)^2}$	$\frac{\partial L}{\partial m} = \sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$	$\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2} = \sum_{i=1}^{3} \frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$m_2 = \frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m} \Big/ \frac{\partial^2 L}{\partial m^2}$
1	1	+.17	-.0700	+ .254667	+ .009840	- 1.120600	- 3.184900	-.066879
	2	-.23		- .164200		- 1.051500		
	3	-.15		- .080520		- 1.012800		
2	1	-.03	+.0533	- .083882	+.016730	- 1.021059	- 3.438830	+.049737
	2	-.15		- .212064		- 1.133055		
	3	+.34		+ .312676		- 1.284716		
3	1	+.67	+.4300	+ .254668	- .177943	- 1.190833	- 5.080876	+.465022
	2	-.11		- .762281		- 2.573777		
	3	+.73		+ .329670		- 1.316266		
4	1	-.54	-.2700	- .291230	+.018882	- 1.248262	- 3.655622	-.264835
	2	-.33		- .060216		- 1.010865		
	3	+.06		+ .370328		- 1.396495		
5	1	-.41	+.0067	- .481688	- .029803	-1.718538	- 4.101439	+.013936
	2	+.11		+ .104445		- 1.032609		
	3	+.32		+ .347440		- 1.350292		
6	1	-.26	+.0967	- .408701	-.024353	-1.479856	- 3.858903	+.103011
	2	+.12		+ .023312		- 1.001630		
	3	+.42		+ .361036		- 1.377417		
7	1	+.54	-.2167	+1.770451	+.853498	-4.424677	- 9.978709	-.131268
	2	-.50		- .308021		-4.277016		
	3	-.69		- .609932		- 1.277016		
8	1	+.19	-.3700	+ .815850	+.207992	-2.218101	- 4.758692	-.326292
	2	-.64		- .291230		- 1.248262		
	3	-.66		- .316628		- 1.292329		
9	1	-.12	-.4433	+ .361036	+ .011911	- 1.377433	- 3.666198	- .440051
	2	-.48		- .036749		- 1.004049		
	3	-.73		- .312376		- 1.284716		
10	1	-.49	-.6533	+ .167774	+ .002661	- 1.083693	- 3.132061	-.652450
	2	-.77		- .118311		- 1.041802		
	3	-.70		- .046802		- 1.006566		
11	1	-.07	-.1367	+ .066896	-.009546	- 1.013446	- 3.308885	-.139555
	2	+.04		+ .182285		- 1.098764		
	3	-.38		- .258727		- 1.196675		
12	1	-.40	-.5267	+ .128735	-.005134	- 1.049476	- 3.182127	-.528283
	2	-.72		- .200836		- 1.119217		
	3	-.46		+ .066967		- 1.013434		
13	1	-.30	-.2633	- .036749	-.387058	- 1.004049	- 4.343556	-.352411
	2	-.65		- .454723		- 1.589322		
	3	+.16		+ .104414		- 1.750185		
14	1	-.14	-.2667	+ .128767	-.010034	-1.049476	- 3.263601	-.269775
	2	-.50		- .246729		- 1.179312		
	3	-.16		+ .107928		- 1.034813		

Sample No.	i	x_i	$m_1 = \bar{x}$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1+(x_i-m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3} \frac{x_i - m}{1+(x_i-m)^2}$	$-\frac{1+(x_i-m)^2}{[1+(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3} -\frac{1+(x_i-m)^2}{[1+(x_i-m)^2]^2}$	m_2 $\frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m}$ / $\frac{\partial^2 L}{\partial m^2}$
15	1	-.09	-.2367	+ .149926	+ .002713	- 1.066950	- 3.177717	- .235846
	2	-.20		+ .036749		- 1.004049		
	3	-.43		- .183962		- 1.106718		
16	1	-.29	+.0333	- .360623	- .021856	- 1.376579	- 3.585193	+.039096
	2	+.12		+ .087663		- 1.022996		
	3	+.27		+ .251104		- 1.185618		
17	1	-.10	-.2400	+ .142798	- .000653	- 1.060774	- 3.131187	-.240209
	2	-.39		- .153452		-1.070113		
	3	-.23		+ .010001		- 1.000300		
18	1	+.26	-.1100	+ .428687	+ .025513	- 1.526159	- 3.878581	-.103422
	2	-.42		- .342958		- 1.341557		
	3	-.17		- .060216		- 1.010865		
19	1	+ .23	- .1367	+ .423670	+ .029076	- 1.514353	- 3.830699	- .129111
	2	- .21		- .073695		- 1.016264		
	3	- .43		- .320905		- 1.300082		
20	1	+ .37	+.3167	+ .053552	- .025787	- 1.008563	- 4.150227	+ .322813
	2	- .09		- .483174		- 1.673112		
	3	+ .67		+ .403835		- 1.468552		
21	1	+ .21	+.0367	+ .178665	- .983165	- 1.094805	-17.060464	- .094328
	2	-.76		-2.181131		-12.252377		
	3	+ .66		+1.019301		- 3.713282		
22	1	- .20	-.0633	- .139303	+ .020958	- 1.057853	- 3.434721	- .057198
	2	+ .23		+ .320905		- 1.300082		
	3	- .22		- .160644		- 1.076786		
23	1	-.44	-.3000	- .142798	+ .084156	- 1.060744	- 4.378292	- .280779
	2	-.62		- .356506		- 1.368375		
	3	+.16		+ .583460		- 1.949243		
24	1	- .28	+.1233	- .481639	- .038750	-1.658197	- 4.063031	- .132837
	2	+.20		+ .077153		-1.017823		
	3	+.45		+ .365736		- 1.387011		
25	1	-.13	-.2367	+ .107928	+ .155433	-1.034813	- 3.564903	-.193096
	2	-.56		- .361036		-1.377417		
	3	-.02		+ .408551		-1.152673		
26	1	+.10	-.0100	+ .111347	- .014051	-1.024632	- 3.322761	-.014229
	2	+.14		+ .153452		-1.070113		
	3	-.27		- .278850		-1.228016		
27	1	-.79	-.3200	- .603260	- .105746	-2.011377	- 4.344870	-.344338
	2	-.09		+ .242846		-1.142660		
	3	-.08		+ .254668		-1.190833		
28	1	-.08	+.1760	- .273953	+ .001564	-1.028078	- 3.244534	-.175518
	2	+.43		+ .271517		-1.216408		
	3	+.18		+ .004000		-1.000048		

Sample No.	i	x_i	$m_1 = \bar{x}$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1-(x_i-m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$	$-\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3} -\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	m_2 $\frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m}$ / $\frac{\partial^2 L}{\partial m^2}$
29	1	+.50		+ .864077		- 2.991572		
	2	-.59	-.0767	- .696923	+.103600	- 2.329134	- 6.332807	-.060341
	3	-.14		- .063554		- 1.012101		
30	1	+.02		- .203830		- 1.123044		
	2	+.17	+.2160	- .046097	+.009519	- 1.006379	- 3.327343	+.213139
	3	+.46		+ .259446		- 1.197929		
31	1	-.35		- .057185		- 1.009800		
	2	-.18	-.2930	+ .114461	+.000091	- 1.039137	- 3.058737	-.292970
	3	-.35		- .057185		- 1.009800		
32	1	+.29		+ .329670		- 1.316266		
	2	- .16	- .0100	- .153452	+.022766	- 1.070113	- 3.456492	-.003414
	3	-.16		- .153452		- 1.070113		
33	1	-.45		+ .017004		- 1.000867		
	2	-.60	-.4667	- .135395	-.000222	- 1.054671	- 3.097560	-.463667
	3	-.35		+ .118623		- 1.042022		
34	1	-.23		- .160644		- 1.076786		
	2	+.21	-.0733	+ .308021	+.018610	- 1.057853	- 3.184115	-.067455
	3	-.20		- .128767		- 1.049476		
35	1	-.43		- .444132		- 1.563278		
	2	-.05	-.0500	+ .000000	+.000000	- 0.000000	- 2.832959	-.050000
	3	+.33		+ .444132		- 1.563278		
36	1	-.62		- .630593		- 2.100061		
	2	-.59	-.1367	- .570533	+6.439597	- 1.909640	-128.928063	-.086753
	3	+.80		+7.640723		-124.918362		
37	1	-.44		- .509956		- 1.734292		
	2	+.50	-.0200	+ .712719	+.101753	- 2.386551	- 5.151350	-.000247
	3	-.12		- .101010		- 1.030507		
38	1	-.14		- .312376		- 1.284716		
	2	+.49	+.1467	+ .387806	+.018548	- 1.436499	- 3.730911	+.130837
	3	+.09		- .056882		- 1.009696		
39	1	-.32		- .689282		- 2.301754		
	2	+.74	+.1900	+ .788530	+.059184	- 2.677252	- 5.983818	-.179792
	3	+.15		- .040064		- 1.004812		
40	1	-.17		+ .013002		- 1.000507		
	2	+.08	-.1833	+ .285611	-.001407	- 1.233970	- 3.497606	-.183402
	3	-.46		- .300020		- 1.263129		
41	1	-.30		- .325339		- 1.308220		
	2	+.07	-.0033	+ .073695	-.016626	- 1.016264	- 3.487431	-.008067
	3	+.22		+ .235018		- 1.162947		
42	1	+.01		+ .167774		- 1.909102		
	2	-.10	-.1533	+ .053451	-.006152	- 1.008563	- 4.070338	-.154811
	3	-.37		- .227577		- 1.153673		

Sample No.	i	x_i	$m_1=\bar{x}$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i-m}{1-(x_i-m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3}\frac{x_i-m}{1-(x_i-m)^2}$	$\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3}\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	m_2 $\frac{\partial^2 L}{\partial m^2}m_1-\frac{\partial L}{\partial m}$ $\frac{\partial^2 L}{\partial m^2}$
	1	+.27		+.283036		−1.234771		
43	2	+.03	+.0067	+.023412	+.010962	−1.001644	−3.549514	+.003512
	3	−.28		−.295486		−1.313099		
	1	−.28		−.070344		−1.014820		
44	2	−.13	−.2100	+.080515	+.000170	−1.019406	−3.034526	−.209944
	3	−.22		−.010001		−1.000300		
	1	−.09		−.146304		−1.063775		
45	2	+.19	+.0533	+.139303	+.000301	−1.057853	−3.121762	+.053396
	3	+.06		+.006700		−1.000134		
	1	−.08		+.104414		−1.032590		
46	2	−.28	−.1833	−.097612	+.000102	−1.028495	−3.061219	−.183267
	3	−.19		−.006700		−1.000134		
	1	−.10		+.212064		−1.133055		
47	2	−.70	−.3033	−.470788	−.057922	−1.630045	−3.882558	−.318219
	3	−.11		+.200802		−1.119458		
	1	+.67		+.745508		−2.509221		
48	2	+.07	+.1367	−.066896	+.082154	−1.013405	−5.512458	+.121697
	3	−.33		−.596458		−1.989832		
	1	−.28		−.077153		−1.017823		
49	2	−.15	−.2033	+.053466	−.000381	−1.008563	−3.028016	−.203426
	3	−.18		+.023312		−1.001630		
	1	−.33		−.320905		−1.300082		
50	2	+.45	−.0367	+.637773	+.116066	−2.123915	−4.543455	−.011154
	3	−.23		−.200802		−1.119458		
	1	+.44		+.883184		−3.074927		
51	2	−.79	−.1433	−1.112848	−.166413	−4.196875	−9.291723	−.160910
	3	−.08		+.063251		−2.019971		
	1	−.15		−.010001		−1.000300		
52	2	+.36	−.1400	+.666667	+.011845	−2.222222	−5.270074	−.137752
	3	−.63		−.644821		−2.147552		
	1	+.09		−.208333		−1.128472		
53	2	+.31	+.2900	+.020008	−.014143	−1.001200	−3.232369	+.290044
	3	+.47		+.186027		−1.102697		
	1	+.21		+.114773		−1.039349		
54	2	−.29	+.0967	−.454693	−.044559	−1.589382	−3.883929	+.108173
	3	+.37		+.295361		−1.255198		
	1	+.39		+.186027		−1.102697		
55	2	−.21	+.2100	−.509956	−.069261	−2.600554	−4.894084	+.224152
	3	+.45		+.254668		−1.190833		
	1	−.30		+.097612		−1.028495		
56	2	−.22	−.3967	+.182394	−.015355	−1.098764	−3.382457	−.401240
	3	−.67		−.295361		−1.255198		

Sample No.	i	x_i	$m_1=\bar{x}$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i-m}{1+(x_i-m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3}\frac{x_i-m}{1+(x_i-m)^2}$	$\frac{1+(x_i-m)^2}{[1+(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3}\frac{1+(x_i-m)^2}{[1+(x_i-m)^2]^2}$	m_2 $\frac{\partial^2 L}{\partial m^2}m_1-\frac{\partial L}{\partial m}\Big/\frac{\partial^2 L}{\partial m^2}$
57	1	−.41	−.1133	− .325339	− .810074	− 1.324459	− 48.783525	−.129916
	2	−.71		− .926626		− 3.371716		
	3	+.78		+4.418971		−44.057350		
58	1	+.04	−.2033	+ .258608	− .120557	− 1.196675	− 4.517265	−.229988
	2	−.69		− .637773		− 2.123915		
	3	+.04		+ .258608		− 1.196675		
59	1	+.43	−.0200	+ .564263	− .009196	− 1.890704	− 4.840247	−.021900
	2	−.48		− .583460		− 1.949243		
	3	−.01		+ .010001		− 1.000300		
60	1	+.69	+ .1433	+ .779753	+ .185625	− 2.643417	− 5.172942	+.107416
	2	−.17		− .347399		− 1.350213		
	3	− .09		− .246729		− 1.179312		
61	1	+.12	- .0400	+ .164203	− .020191	− 1.080198	− 3.316004	−.046089
	2	+.09		+ .132234		− 1.052162		
	3	−.33		− .316628		− 1.183644		
62	1	+.63	+ .3467	+ .308021	− .004016	− 1.277016	− 3.585766	+.347820
	2	+.36		+ .013302		− 1.000530		
	3	+.05		− .325339		− 1.308220		
63	1	−.00	+ .0967	− .097509	+ .163267	- 1.028435	− 4.770116	+.062373
	2	−.40		− .659155		− 1.654673		
	3	+.69		+ .915915		− 2.087008		
64	1	+.13	+ .0567	+ .073695	+ .247951	− 1.016264	−13.818279	+.038756
	2	−.69		−1.057383		− 7.938848		
	3	+.73		+1.231645		− 4.863167		
65	1	−.39	−.3200	− .070344	+ .005018	− 1.014820	− 3.174239	−.318419
	2	−.13		+ .197115		− 1.115161		
	3	−.44		− .121753		− 1.044258		
66	1	−.12	+ .2267	− .394067	− .022492	− 1.447201	− 3.714500	+.232755
	2	+.51		+ .308021		− 1.255198		
	3	+.29		+ .063554		- 1.012101		
67	1	−.16	+.0267	− .193442	+.000660	− 1.110956	− 3.230548	+.026496
	2	+.22		+ .200802		− 1.119458		
	3	+.02		− .006700		- 1.000134		
68	1	-.51	-.3833	−.129081	+.003074	- 1.049717	− 3.165866	-.382029
	2	-.44		−.057185		- 1.009800		
	3	-.20		+ .189340		1.106349		
69	1	+.35	+.3133	+ .036749	−.001929	− 1.004049	− 3.119208	+.321050
	2	+.43		+ .118311		- 1.041802		
	3	+.16		− .156989		- 1.073357		
70	1	+.59	+.2533	+ .379751	+ .029960	− 1.416284	− 3.618580	+.245021
	2	+.03		− .235018		− 1.162947		
	3	+.14		−.114773		− 1.039349		

Sample No	i	x_i	$m_1=\bar{x}$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i-m}{1+(x_i-m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3}\frac{x_i-m}{1+(x_i-m)^2}$	$\frac{1+(x_i-m)^2}{[1+(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3}\frac{1+(x_i-m)^2}{[1+(x_i-m)^2]^2}$	m_2 $\frac{\partial^2 L}{\partial m^2}\,m_1-\frac{\partial L}{\partial m}\Big/\frac{\partial^2 L}{\partial m^2}$
71	1	−.19	−.2300	+ .040064	− .002041	− 1.004812	− 3.111969	−.230656
	2	−.38		− .153452		− 1.070113		
	3	−.12		+ .111347		− 1.037044		
72	1	−.45	−.1667	− .307638	− .000618	− 1.276344	− 3.539620	−.167175
	2	−.16		+ .007000		− 1.000147		
	3	+.11		+ .300020		− 1.263129		
73	1	+.71	+.0533	+1.155256	+ .284473	− 4.424677	− 7.979731	+.017651
	2	−.48		− .745580		− 2.508265		
	3	−.07		− .125203		− 1.046789		
74	1	−.33	−.1167	− .223467	+ .010204	− 1.147540	− 3.407557	−.113705
	2	+.15		+ .287122		− 1.251454		
	3	−.17		− .053451		− 1.008563		
75	1	−.19	−.1167	− .073695	+ .025011	− 1.016264	− 2.718663	−.109974
	2	−.39		− .295361		− 1.255198		
	3	+.23		+ .394067		− 1.447201		
76	1	−.01	−.3367	+ .365736	− .057547	− 1.387011	− 4.016400	−.351028
	2	−.73		− .465270		− 1.615943		
	3	−.27		+ .066998		− 1.013446		
77	1	−.63	−.0683	− .820607	+ .089560	− 2.707683	− 7.103430	−.055692
	2	−.09		− .021710		− 1.001413		
	3	+.53		+ .931877		− 3.294334		
78	1	−.04	−.0600	+ .020008	− .058810	− 1.001200	− 6.266361	−.069385
	2	+.47		+ .737032		− 2.477060		
	3	−.61		− .815850		− 2.788101		
79	1	+.39	+.1900	+ .208333	+ .314714	− 1.128472	− 3.268264	+.093706
	2	+.38		+ .197115		− 1.115161		
	3	+.10		− .090734		− 1.024631		
80	1	−.77	−.2600	− .689282	− .095425	− 2.301754	− 5.020126	−.279008
	2	−.16		+ .101010		− 1.030507		
	3	+.15		+ .492847		− 1.687865		
81	1	−.03	−.2267	+ .204504	− .066846	− 1.123849	− 3.979725	−.243397
	2	−.01		+ .227262		− 1.152521		
	3	−.64		− .498612		− 1.703355		
82	1	+.21	+.4700	− .278850	− .004174	− 1.228016	− 3.420049	+.364611
	2	+.71		+ .254668		− 1.190833		
	3	+.49		+ .020008		− 1.001200		
83	1	−.26	−.2067	− .053451	+ .001621	− 1.008563	− 3.074303	−.206173
	2	−.28		− .073695		− 1.016264		
	3	−.08		+ .128767		− 1.049476		
84	1	−.16	−.0933	− .066998	+ .019230	− 1.013446	− 3.292881	−.087460
	2	+.23		+ .361036		− 1.057853		
	3	−.35		− .274808		− 1.221582		
85	1	−.31	+.0900	− .476190	+ .021325	− 1.643990	− 4.430139	+.085186
	2	+.06		− .030027		− 1.002704		
	3	+.52		+ .527542		− 1.783445		

Sample No.	i	x_i	$m_1 = \bar{x} = \frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1-(x_i-m)^2}$	$\frac{\partial L}{\partial m} = \sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$	$\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2} = \sum_{i=1}^{3} -\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$m_2 = \frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m} \Big/ \frac{\partial^2 L}{\partial m^2}$
86	1	+.36	+.1233	+ .250748	− .108436	− 1.185102	− 3.533041	+.153992
	2	+.36		+ .250748		− 1.185102		
	3	−.35		− .609932		− 1.162837		
87	1	+.56	−.1567	+1.473657	+ .626575	− 6.399500	− 9.508069	−.090801
	2	−.60		− .551721		− 1.853371		
	3	−.43		− .295361		− 1.255198		
88	1	−.45	−.1833	− .287122	+ .038693	− 1.241454	− 3.814450	−.173156
	2	−.29		− .107928		− 1.034813		
	3	+.19		+ .433743		− 1.538183		
89	1	−.09	−.0800	− .010001	+ .005768	− 1.000300	− 3.710134	−.078445
	2	−.39		− .341851		− 1.341559		
	3	+.24		+ .357620		− 1.368275		
90	1	−.09	−.2500	+ .164203	− .002885	− 1.080198	− 3.198063	−.250902
	2	−.44		− .197115		− 1.115161		
	3	−.22		+ .030027		− 1.002704		
91	1	+.24	+.0800	+ .164203	+ .001397	− 1.080198	− 3.142172	+.079555
	2	+.06		− .020008		− 1.001200		
	3	−.06		− .142798		− 1.060774		
92	1	+.05	−.0867	+ .139303	− .034122	− 1.057853	− 3.632231	−.096094
	2	+.12		+ .215139		− 1.137879		
	3	−.43		− .389164		− 1.436499		
93	1	−.10	+.0833	− .189672	+ .007625	− 1.106718	− 3.300383	+.080990
	2	+.03		− .053451		− 1.008563		
	3	+.32		+ .250748		− 1.185102		
94	1	−.08	+.1233	− .212064	− .005763	− 1.133055	− 3.204354	+.125098
	2	+.26		+ .139303		− 1.057853		
	3	+.19		+ .066998		− 1.013446		
95	1	−.40	−.2233	− .183494	+ .143083	− 1.098764	− 4.961730	−.194463
	2	+.30		+ .720642		− 2.415765		
	3	−.57		− .394067		− 1.447201		
96	1	+.01	−.0700	+ .080515	+ .000291	− 1.019406	− 3.031471	−.069904
	2	−.13		− .060216		− 1.010865		
	3	−.09		− .020008		− 1.001200		
97	1	−.04	+.0433	− .083882	+ .009877	− 1.021059	− 3.279518	+.040288
	2	+.28		+ .250748		− 1.185102		
	3	−.11		− .156989		− 1.073357		
98	1	−.21	−.3067	+ .097612	− .018516	− 1.028495	− 3.240918	−.312413
	2	−.18		+ .128767		− 1.049476		
	3	−.53		− .235018		− 1.162947		
99	1	−.02	−.0700	+ .050125	+ .000000	− 1.007531	− 2.015062	−.070000
	2	−.12		− .050125		− 1.007531		
	3	−.07		+ .000000		− 0.000000		
100	1	−.09	−.0600	− .030027	+ .000000	− 1.032758	− 2.065516	−.060000
	2	−.06		+ .000000		− 0.000000		
	3	−.03		+ .030027		− 1.032758		

In order to analyze the results of the calculations on the preceding ten pages it is necessary that we find the theoretical variance for $\bar{x}$ and $\hat{m}$ from the formulae derived for this purpose ·lier in this paper.

From (7) page 90, we get after setting $a=1$, $\rho=2$, and $n=3$,

$$\sigma^2_{\bar{x}} = \frac{1}{21} = .047619 \tag{10}$$

and from (8) page 90 after similar substitutions

$$\sigma^2_{\hat{m}} = \frac{1}{30} = .033333 \tag{11}$$

From (9) page 90 we get for the efficiency of the mean when

$$\rho = 2 \quad E = .70 \tag{12}$$

Now by actual calculation from the ungrouped data the mean square deviation from zero which we shall designate as

$$\frac{\Sigma \bar{x}^2}{N} = .048611 \tag{13}$$

and

$$\frac{\Sigma m_2^2}{N} = .047612 \tag{14}$$

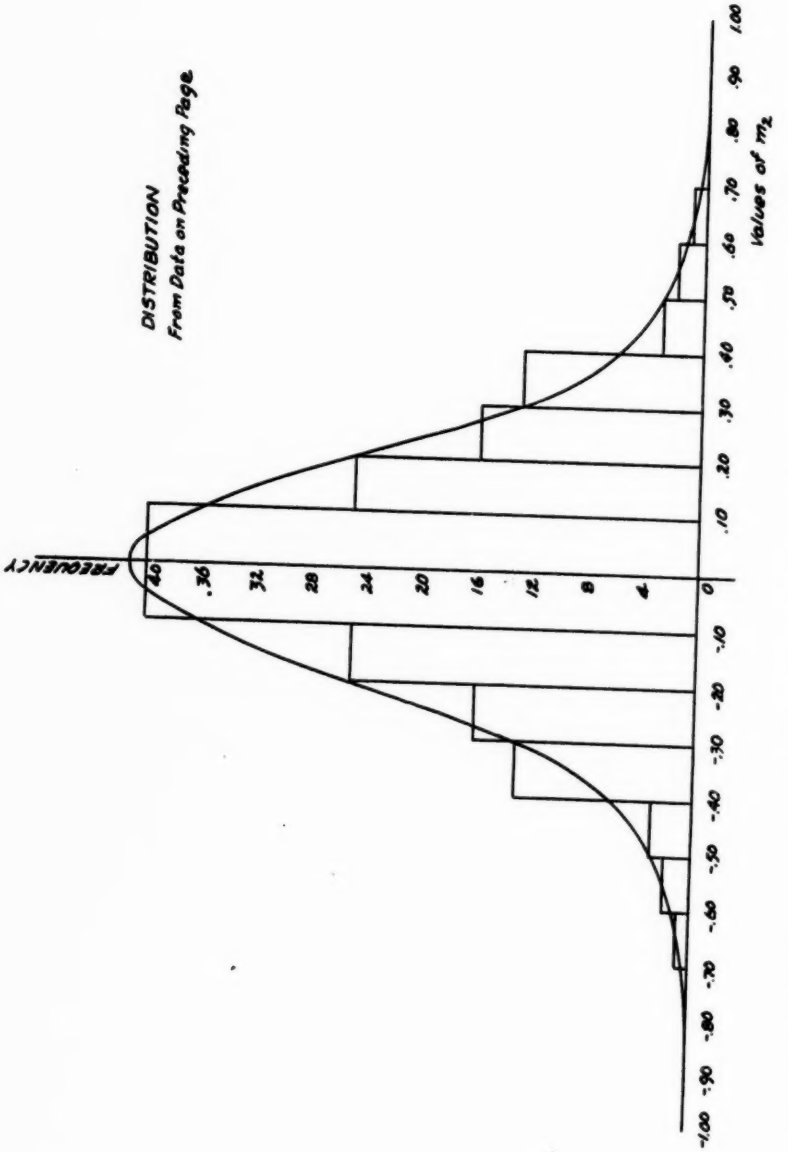
Comparing (10) and (13) it is evident that such a difference can be said to be well within the limits of random sampling. The difference between (11) and (14) is of such magnitude that we can not say that it might be expected in the course of random sampling. Now since m_2 is our estimate of $\hat{m}$ it is evident that either a single approximation by Newton's method is not adequate to give the best results or the approximation,

$$\sigma^2_{\hat{m}} = \frac{a^2(\rho-1)}{np(2\rho+1)},$$

to the variance is not valid in the case of small samples. That the latter seems to be the case the writer firmly believes. The reason for this belief lies in the fact that in a subsequent case a sample of 3 was examined by Newton's method, and starting with $m_1 = \bar{x} = -.07$ the values $m_2 = -.066879$ and $m_3 = -.066791$ were obtained. There is not sufficient improvement here to cause one to suppose that by taking a third approximation we would obtain a variance in keeping with the one derived from theory. Also considering the mean square deviation from zero for $\bar{x}$ and m_2 it seems that the gain in accuracy to be expected from the use of the method of maximum likelihood solution instead of the arithmetic mean is not sufficiently great, in the case of samples of three, to warrant the additional labor involved in calculation. We must be sure, however, that in using the arithmetic mean we are using an approximation to m which complies with the qualifications given on pages 95 and 96. A graph of the distribution as found from the calculations is given on page 106. The histogram represents the grouped data while the smoothed curve is a rough approximation to the actual form of the distribution.

Histogram Data

	m_2	+	−	Totals
	0.00	2	2	0
.01	−.10	16	20	40
.11	.20	8	17	25
.21	.30	4	12	16
.31	.40	4	9	13
.41	.50	1	2	3
.51	.60	1	1	2
.61	.70	0	1	1
.71	.80	0	0	0
.81	.90	0	0	0
.91	1.00	0	0	0
		36	64	100


DISTRIBUTION
From Data on Preceding Page
FREQUENCY
40
36
32
28
24
20
16
12
8
4
0
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.00
-.10
-.20
-.30
-.40
-.50
-.60
-.70
-.80
-.90
-1.00
Values of m_2

The totals have been used in the histogram which has been forced to be symmetrical so as to give the effect of a sample of twice the size.

The tabulation of the histogram data draws attention to the great excess of samples having negative values for $\bar{x}$ and m_2. This has caused the writer no little concern. In examining the signs of the observations we note that there are 183 – and only 117 + values. Assuming + and – values to be equally likely this gives a deviation from 150 of 33 or 3.81 times its standard error which is incredible. It seems therefore that Tippett's numbers are not random in this respect, and that it perhaps would have been better to toss a coin to determine the signs.

As a final check and in an effort to place the type of the distribution the value of β_2 has been calculated, and found to be

$$\beta_2 = 3.056$$

β_1 was not calculated as the excess of negative signs would lead to an erroneous value. There is every reason to believe that β_1 should be zero. These facts suggest that the curve is very near to the normal curve, but perhaps slightly more leptokurtic. But, why, if this is the case, there is not better agreement between (11) and (14) page 104 the writer is unable at the present to state.

John L. Carlson

A NEW THEORY OF DEPRECIATION OF PHYSICAL ASSETS

By Robert E. Moritz

1. The various methods employed for the computation of the depreciation of a physical asset are as many devices for recovering, by means of a yearly charge to production during the life-time of the asset, its reduction in value. The methods differ according to the answers given to such questions as the following: Should the yearly charge be based on original cost or on replacement value? Should the yearly charge be uniform over the life-time of the asset? If not uniform, should the depreciation charge be proportional to the actual reduction in market value, or, in the case of new plant, should such charges be minimized or wholely deferred during the earlier years when the plant is trying to establish itself? Should interest be disregarded, or should the yearly charges to production be accumulated with interest? If interest is to be considered, should the rate be the effective rate on the capital employed in the business, or the commercial rate? All these questions and others have received careful consideration.[1]

Quite as important as those already mentioned are two other considerations, which have been generally ignored or overlooked. There is first the rather obvious fact that depreciation is inseparably tied up with the question of repairs. Depreciation is greatest when the asset is new, when repair charges are negligible, and it diminishes as repair costs grow. The problem of depreciation, therefore, cannot be adequately treated aside from repairs. Within certain limits depreciation may be compensated by repairs. The yearly depreciation charge to production, therefore, should not be based on original cost or renewal cost alone, but on original cost plus costs of repairs during the life-time of the asset.

In the second place, the life-time of an asset is not a constant

[1]See Saliers, E. A., Depreciation, Principles and Applications, New York (1922).

as is generally assumed, but is a variable which, like depreciation itself, is definitely related to the repair function. Aside from obsolescence, the value of an asset could be kept practically intact indefinitely by sufficiently increasing the outlay for repairs. There is always a threshold period of time when it is a question whether to scrap or to continue to repair, and frequently this threshold extends over a period of years. In short, the life-time of an asset is generally an unknown quantity, the determination of which requires the solution of an equation which expresses the condition that the annual charge to production, necessary to recover the original cost of an asset together with all repair costs, shall be a minimum.[2]

The present paper is an attempt to treat the problem of depreciation from the point of view here suggested. The problem, then, is to determine the life-time of an asset such that the annual charge to production, necessary to cover original cost and all repair charges, shall be a minimum.

2. Let us denote the original cost of the asset by C, the cost of repairs during the first x years by $R(x)$, then the total outlay to be recovered is $C + R(x)$. Furthermore, let $U(x)$ denote the average yearly charge to production necessary to recover the total outlay in x years. Then, disregarding for the present all interest considerations, we have

$$U(x) = \frac{C + R(x)}{x} \tag{1}$$

The notation $U(x)$ suggests that, in general, the unit charge to production will be a function of x.

We shall now define the life-time of an asset as that value of x which will render the value of $U(x)$ in (1) a minimum.

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[2] J. S. Taylor, A Statistical Theory of Depreciation, Journal of the American Statistical Association, Vol. 18 (1923), p. 1010, is, I believe, the first writer who recognized in part the principle here set forth. He calls attention to the fact that the useful life of a machine depends both on the manner of distributing depreciation charges and on the assumed interest rate.

The analytical conditions that a function of x may have a minimum are that the first derivative of the function with respect to x be zero and that the second derivative of the function with respect to x be positive. If, as is customary, we denote the first and second derivatives of $U(x)$ by $U'(x)$ and $U''(x)$ respectively, and those of $R(x)$ by $R'(x)$ and $R''(x)$ respectively, we find on differentiating

$$U'(x) = \frac{xR'(x) - C - R(x)}{x^2},$$

$$U''(x) = \frac{x^2R''(x) - 2xR'(x) + 2R(x) + 2C}{x^3} = \frac{R''(x) - 2U'(x)}{x},$$

from which it is evident that the life-time of an asset must satisfy the two conditions

$$xR'(x) - C - R(x) = 0, \qquad R''(x) > 0 \tag{2}$$

In short, the life-time of an asset is given by that root of the equation $xR'(x) - R(x) = C$ which will make $R''(x) > 0$.

For example, let us suppose that the repair function is given by the equation $R(x) = ax^2 + bx + c$. Then $R'(x) = 2ax + b$, $R''(x) = 2a$, and the conditions (2) reduce to

$$ax^2 = C + c, \qquad a > 0$$

The life-time of the asset is therefore equal to $\sqrt{(C+c)/a}$ provided the coefficient a is positive. It is interesting to observe that x is independent of the constant b.

3. In the preceding discussion no allowance was made for the salvage value of the asset. Let us denote the scrap-value of the asset after x years by $S(x)$, then the average yearly de-

preciation, interest again not considered, is

$$(3) \qquad U(x) = \frac{C + R(x) - S(x)}{x}.$$

and the conditions which will make $U(x)$ a minimum are

$$(4) \quad x\left[R'(x) - S'(x)\right] - \left[C + R(x) - S(x)\right] = 0, \quad R''(x) > S''(x).$$

If the scrap-value is a constant, both $S'(x)$ and $S''(x)$ vanish, and the life-time of the asset is determined by

$$(5) \qquad x R'(x) - C - R(x) + S(x) = 0, \quad R''(x) > 0.$$

The conditions (2) include (4) if we replace $R(x)$ by $R(x) - S(x)$, that is, if in the outset we diminish the repair function by the salvage value at time x ; to include (5) it is sufficient to replace C, the original cost of the asset, by $C - S$, the difference between original cost and scrap value. With these modifications we may treat (2) as representing the general case.

4. To avoid any possible confusion, let us denote by $T(x)$ the total outlay to be recovered by uniform annual charges to production during the life-time of the asset. Taking account of the residual value $S(x)$ we see that

$$(6) \quad T(x) = C + R(x) - S(x), \qquad T'(x) = R'(x) - S'(x)$$

and (3) and (4) take the simpler forms

$$(7) \quad U(x) = T(x)/x, \qquad x T'(x) - T(x) = 0, \quad R''(x) > S''(x).$$

From the first and second of the equations (7) follows:

$$(8) \qquad T'(x) = U(x)$$

which may be appropriately called the life-equation of an asset since its solution yields the life-time of the asset as defined in 1.

5. When the repair function and the salvage function are known, the real roots of the life-equation may be found either by direct methods or by methods of approximation. However, in the great majority of cases which occur in practice the value of $T(x)$ is given only empirically, from the recorded experience relating to the asset in question, and the data available may not lend itself to analytical treatment. In all such cases the life-time of the asset may be determined approximately by means of the following simple graphic method.

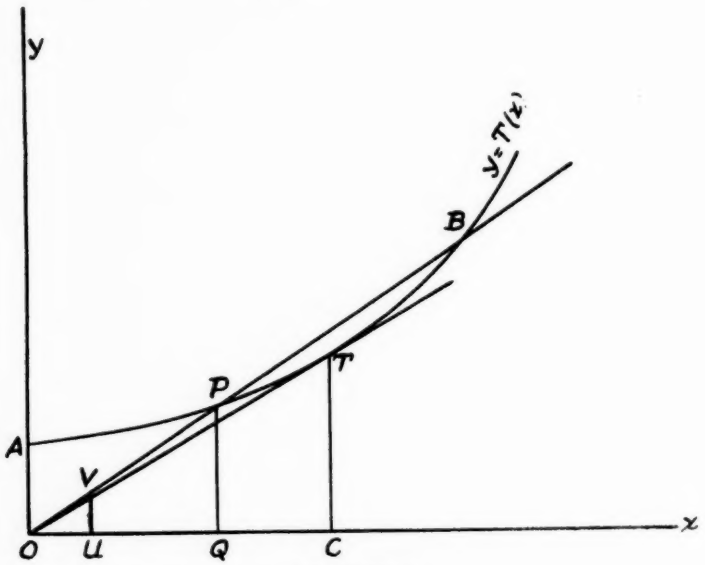


FIGURE 1.

Let AB (Fig. 1), represent the graph of the equation $y = T(x)$, constructed in Cartesian coordinates. We shall call it the total outlay graph, because the ordinate y of any point (x, y) on this graph represents the total outlay during the period of x years if the asset were scrapped at the end of this period. The straight line, OP, joining the origin O to any point P on AB, we shall call the uniform charge to production graph. It enables us to determine at sight the aggregate amount that must

be charged to production during any given period of time in order to recover the total outlay OP for the time x on the basis of uniform distribution over the entire period x. If x is expressed in years, the ordinate UV, of the point on OP whose abscissa is unity, will represent the uniform charge to production per year, which is required to recover the total outlay for x years.

Now it is obvious that this unit charge UV will vary with the slope of the line OP. It will be least when the slope is least, that is to say, when the point P is such that the line OP is tangent to the total outlay graph. The abscissa, OC, of the point of contact, T, is then the life-time of the asset under consideration.

To determine the life-time of an asset, interest considerations being disregarded, we need therefore only construct the total outlay graph AB, then draw the tangent OT, and finally measure the abscissa of the point of contact T.

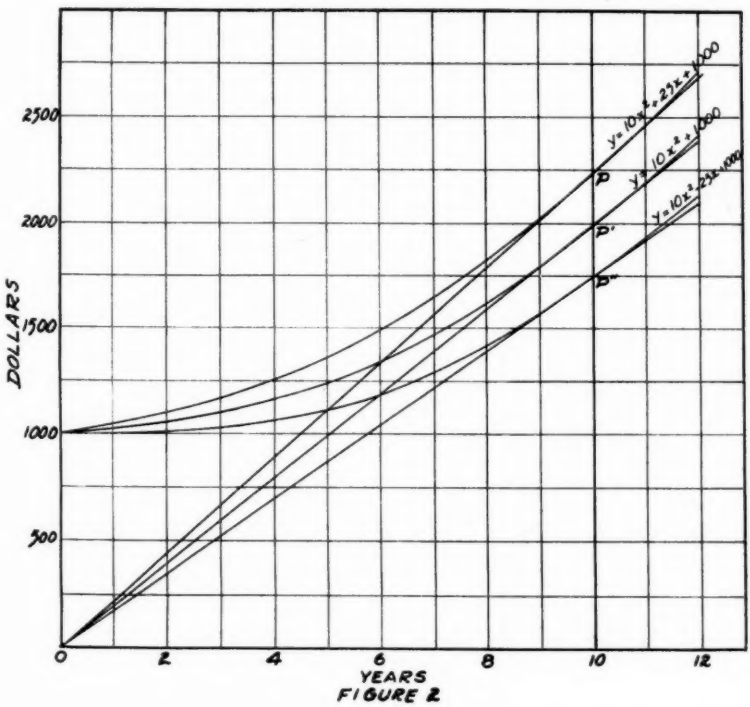


FIGURE 2

(Fig. 2) shows the construction when $T(x)$ has the forms

$$10x^2 + 25x + 1000,$$
$$10x^2 + 1000, \text{ and}$$
$$10x^2 - 25x + 1000$$

respectively. In each case the life-time is found to be 10 years, which verifies the theoretical conclusion of 2: that the life-time is independent of the coefficient of x.

We have seen from graphical considerations that the uniform charge to production will be a minimum when its graph is tangent to the total outlay graph. This condition is precisely the condition expressed by equation (8), which asserts that when x is the life-time of the asset $T'(x)$, the slope of the tangent to the total outlay graph, is numerically the same as the yearly charge to production.

6. We now come to consider the problem of finding the life-time of an asset when interest at a specified rate is to be taken into account. In this case, the various items that make up the total outlay, as well as the component charges to production, must be replaced by their present values at some arbitrarily chosen epoch, as say, the epoch zero.

Let us attempt an analytical solution of the problem. Let Δt represent a small interval of time. The outlay during the interval from t to $t+\Delta t$ is $T(t+\Delta t) - T(t)$. If the specified rate of interest is, i, and if we represent the discount factor by $1/(1+i)$ the conventional symbol v, then the present value of

$$T(t+\Delta t) - T(t)$$

at the epoch 0 has some value between $[T(t+\Delta t) - T(t)]v^t$ and $[T(t+\Delta t) - T(t)]v^{t+\Delta t}$, let us say

$$[T(t+\Delta t) - T(t)]v^{t+\theta\Delta t}$$

where θ has some value between 0 and 1. The total outlay during the time t, evaluated for the epoch 0, is therefore

$$C + \Sigma[T(t+\Delta t) - T(t)]v^{t+\theta\Delta t} \tag{9}$$

the sum extending over all the time intervals between 0 and t.

Now $v^t > v^{t+\theta.\Delta t} > v^{t+\theta_M \cdot \Delta t}$ where θ_M is the greatest among all the fractions θ. We have, therefore,

$$\left[T(t+\Delta t)-T(t)\right]v^t > \left[T(t+\Delta t)-T(t)\right]v^{t+\theta\cdot\Delta t}$$

(10)

$$> \left[T(t+\Delta t)-T(t)\right]v^{t+\theta_M\cdot\Delta t}$$

If the intervals Δt are all equal and their number n, then $\Delta t = t/n$, and as n is increased indefinitely Δt approaches 0. Then $t+\theta_M.\Delta t$ approaches t, and we see from (10) that (9) must have the same limit as

$$C+\sum\left[T(t+\Delta t)-T(t)\right]v^t \tag{11}$$

To determine this limit we write

$$T(t+\Delta t)-T(t) = \frac{T(t+\Delta t)-T(t)}{\Delta t}\,\Delta t$$

where the first factor on the right represents the difference quotient which approaches $T'(t)$ as a limit as Δt approaches 0 as a limit. With this relation introduced into (11), we obtain for the present value at the epoch 0 of all the increments of outlay during the time t, the intervals of time being infinitesimal,

$$\overline{\overline{T}}(t) = C + \underset{\Delta t \to 0}{Limit}\left[\sum \frac{T(t+\Delta t)-T(t)}{\Delta t}\cdot v^t.\Delta t\right] = \tag{12}$$

$$C+\int_0^t v^t.\,T'(t)\cdot dt.$$

In a like manner we may derive an expression for $\bar{D}(t)$, the limit of the sum of the present values at epoch 0 of all the

charges to production during the time t apportioned at some uniform rate U to each of the intervals Δ. . The charge apportioned to the interval from t to $t+\Delta t$ is $U.\Delta t$, its present value at epoch O is $U.v^{t+\theta.\Delta t}.\Delta t$. The present value of the sum of these amounts for all the intervals Δt between O and t is

$$\Sigma\, U.v^{t+\theta.\Delta t}.\Delta t$$

which for infinitesimal values of Δt has the same limit as $\Sigma\, U.v^{t}.\Delta t$, so that finally

$$\bar{D}(t) = \underset{\Delta t \to O}{Limit}\, \Sigma\, U.v^{t}.\Delta t = U\int_{o}^{t} v^{t}.dt. \tag{13}$$

Let $U(x)$ be the value which must be assigned to U in order to recover $\bar{T}(x)$, the total outlay for x years through a uniform charge to production, interest considered. Then $\bar{D}(x)$ must equal $\bar{T}(x)$, that is,

$$U(x)\cdot\int_{o}^{x} v^{t}.dt = C+\int_{o}^{x} v^{t}.T'(t).dt,$$

from which

$$U(x) = \frac{C+\int_{o}^{x} v^{t}.T'(t).dt}{\int_{o}^{x} v^{t}.dt} \tag{14}$$

The life-time of the asset is that value of x in (14) which will make $U(x)$ a minimum. The derivative of $U(x)$ with respect to x must therefore vanish. Differentiating (14) with respect to x and setting the result equal to O, we find

$$v^{x}.T'(x)\int_{o}^{x} v^{t}.dt - \left[C+\int_{o}^{x} v^{t}.T'(t).dt\right] v^{x} = O,$$

from which

$$T'(x) = \frac{C+\int_{o}^{x} v^{t}.T'(t).dt}{\int_{o}^{x} v^{t}.dt}, \tag{15}$$

which is the life-equation of the asset, interest considered.

7. In deriving equation (13) we apportioned the charges to production for an interval Δt and found the sum of the present values at epoch O. $\bar{D}(t)$ is the limiting value of this sum as the intervals Δt are indefinitely diminished. If, as is customary, no charge is made to production until the end of the year, this single charge will be the aggregate amount of the constituent portions for the separate intervals Δt, accumulated with interest to the end of the year. The charge for the interval from t to $t+\Delta t$ is $U.\Delta t$, its amount at rate i to the end of the year is $U.(1+i)^t.\Delta t$, where t is the time to the end of the year, and the equivalent single charge at the end of the year is

$$\bar{U} = \lim_{\Delta t \to 0} \sum U.(1+i)^t.\Delta t = U. \lim_{\Delta t \to 0} \Sigma (1+i)^t.\Delta t$$

$$= U \int_0^i (1+i)^t.\,dt = \frac{iU}{\log(1+i)}. \tag{16}$$

8. As an example let us again take $T(t) = at^2+bt+c$, then

$$T'(t) = 2at+b \int_0^t v^t dt = v^t \log v \Big|_0^t = (v^t-1)/\log v,$$

$$\int_0^t v^t.\,T(t).\,dt = \int_0^t v^t(2at+b)\,dt.$$

$$= \left[2atv^t+b(v^t-1)\right]/\log v - 2a(v^t-1)/(\log v)^2$$

$$\bar{T}(t) = c+\left[2atv^t+b(v^t-1)\right]/\log v - 2a(v^t-1)/(\log v)^2, \tag{17}$$

$$\bar{D}(t) = U.(v^t-1)/\log v, \tag{18}$$

and (15) reduces to

$$v^x \log v^x = 1 + \frac{c(\log v)^2}{2a}. \tag{19}$$

While the life-equation (19) cannot be solved algebraically, it is evident that an approximate solution for could be obtained from a list of tabulated values of the function $v^x \log v^x$. When such a table is not available, an approximate solution to any desired degree of accuracy may be obtained as follows:

We may write for v, $e^{\log v}$ where e is the base of the natural system of logarithms. (19) then takes the form

$$e^{x \log v} - x \log v = 1 + c(\log v)^2/2a. \tag{20}$$

On expanding the first term of this equation into a power series in x, and simplifying the result, we have

$$x^2 + x^3(\log v)/3 + x^4(\log v)^2/12 + x(\log v)^3/60 + \cdots = c/a,$$

whence

$$x^2 = \frac{c/a}{1 + x(\log v)/3 + x^2(\log v)^2/12 + x^3(\log v)^3/60 + \cdots}.$$

Now for all ordinary rates of interest $\log v$ is necessarily very small, so that if we denote successive approximations of x by x_1, x_2, x_3, etc.,

$$x_1 = (c/a)^{1/2}, \quad x = \left[\frac{c/a}{1 + x_1(\log v)/3}\right]^{1/2},$$

$$x_3 = \left[\frac{c/a}{1 + x_2(\log v)/3 + x_2^2(\log v)^2/12}\right]^{1/2}$$

$$x_4 = \left[\frac{c/a}{1 + x_3(\log v)/3 + x_3^2(\log v)^2/12 + x_3^3(\log v)^3/60}\right]^{1/2}$$

, etc.

Let us take the special case, previously considered in 5., when $c = 1000$, $b = 0$, $a = 10$, and let the assumed rate of interest be 6 percent. Then

$$\log v = -0.058\,269, \quad (\log v)^2 = 0.003\,395,$$

$$1/\log v = -17.161\,788, \quad 1/(\log v)^2 = 294.526\,967,$$

and we find

$$T(t) = 10t^2 + 1000,$$

$$\bar{T}(t) = 1000 - 343.24tv^t - 5890.54(v^t - 1),$$

$$\bar{D}(t) = 17.162\,U(1 - v^t),$$

and the life-equation is

$$v^x - \log v^x = 1.169\,764$$

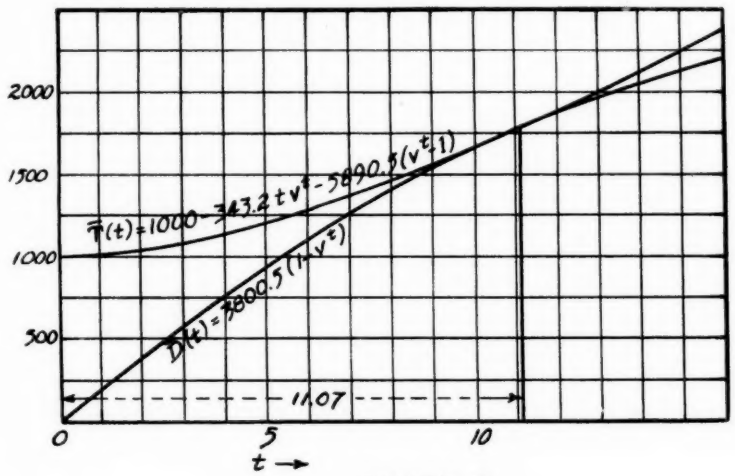


FIGURE 3

The first four successive approximations for x give

$$x_1 = 10, \quad x_2 = 11.14, \quad x_3 = 11.05$$

$$x_4 = 11.07$$

The value $x = 11.07$ substituted in (14) and (16) give us

$$U(x) = 221.45 \quad \text{and} \quad \bar{U}(x) = 228.03$$

This value of $U(x)$ substituted for U in the expression for $\bar{D}(t)$ gives

$$\bar{D}(t) = 3800.5(1 - v^t),$$

which represents the present value at epoch 0 of the aggregate momentary charges during a period t at a rate such as to recover the total outlay 11.07 years, the theoretical life-time of the asset. The momentary rate is 221.45 per year, the equivalent single charge to production at the end of each year is 228.03.

(Fig. 3) shows the graphs of the two equations.

$$\bar{T}(t) = 1000 - 343.24\, t v^t - 5890$$

and

$$\bar{D}(t) = 3800.5(1 - v^t).$$

The abscissa of the common ordinate of the two curves represents the life-time of the asset.

9. It appears from (Fig. 3) that at the point common to the two graphs, the graphs have a common tangent as well as a common ordinate. To see whether or not this is a general property let us trace the changes in the total outlay and total charge to production functions when interest is taken into account.

In the first place it is evident that the increments of the ordinates of both of the graphs in (Fig. 1) must be replaced by their present values at the chosen epoch. If this epoch is the effect in question will be to shorten progressively the ordinates of both graphs. The charge to production graph will then be no longer a straight line but some convex curve, while the total outlay graph will go over into another graph which is less concave than the original graph. But both graphs will continue to rise indefinitely as we proceed from left to right because the increments of their ordinates, while decreasing indefinitely remain positive.

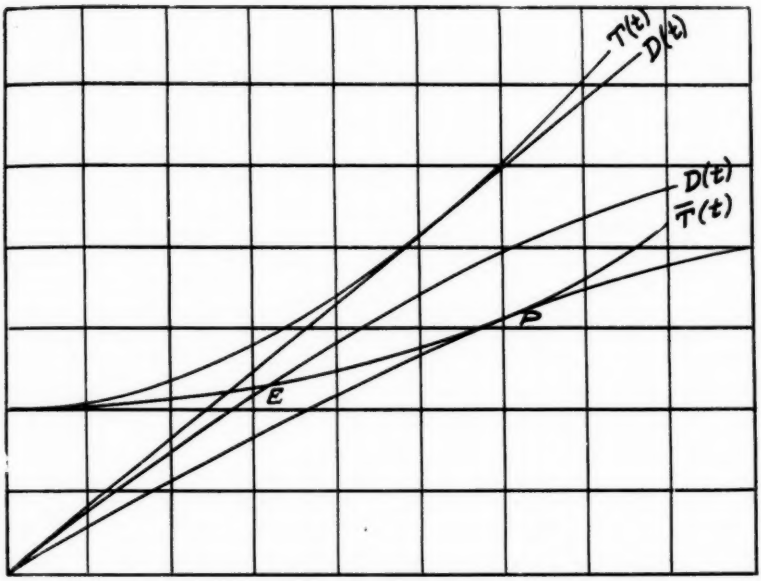


FIG. 4

In (Fig. 4) let $T(t)$ and $D(t)$ represent respectively the total outlay graph and the charge to production graph, interest disregarded. $\bar{T}(t)$ the total outlay graph, interest considered, and $\bar{D}(t)$ the charge to production graph, interest considered, through any point E on $\bar{T}(t)$. The ordinates on $\bar{D}(t)$ represent the present values at epoch O of the momentary charges to production during time t at a rate such as to recover the entire outlay during the time corresponding to the abscissa of the point E. This rate is measured by the initial slope of $\bar{D}(t)$, the slope of $\bar{D}(t)$ when $t=O$.

Let us follow the changes in this slope for the various positions of the point E as it moves along $\bar{T}(t)$ from left to right. It is evident that this slope at first decreases, also that it cannot keep on decreasing indefinitely, it is therefore plausible that it will ultimately increase, reaching a minimum value at the point P where the $\bar{T}(t)$ curve and the $\bar{D}(t)$ curve have a common tangent. The abscissa of the point of contact, P, is then the lifetime of the asset under discussion.

10. The foregoing considerations, however plausible, are open to objections, because we have reasoned from graphs resulting from the assumption of a special law governing the repair function. Different assumptions might give rise to essentially different graphs. We shall, therefore establish the conclusions above arrived at, by an analytical proof, which is independent of any assumptions regarding the nature of the outlay function. We shall prove the

Theorem: If the rate U of a uniform charge to production curve is a minimum, this curve is tangent to the corresponding total outlay curve, and the abscissa of the point of contact represents the life-time of the asset. Conversely,

If a uniform charge to production curve is tangent to the corresponding total outlay curve, U is a minimum.

To prove this theorem, let $y = \bar{T}(t)$ be the equation of the total outlay curve, $y = \bar{D}(t) = U\int_0^t v^t dt$, the equation of the uniform charge to production curve, and x the abscissa of a point common to the two curves.

Then $\bar{T}(x) = \bar{D}(x) = U\int_0^x v^t dt$ from which

$$U = \bar{T}(x) \Big/ \int_0^x v^t dt. \tag{21}$$

Since by hypothesis U is a minimum, its derivative with respect to x must vanish, that is

$$\bar{T}'(x)\int_0^x v^t dt - v^x \bar{T}(x) = 0 \tag{22}$$

From (22) and (21) follows

$$\bar{T}'(x) = v^x \bar{T}(x) \Big/ \int_0^x v^t dt = v^x U = \bar{D}'(x). \tag{23}$$

This shows that at the point common to the two curves their slopes are equal, they have therefore a common tangent, and since

U has a minimum value, x must represent the life-time of the asset.

To prove the converse theorem we observe that if the two curves have a common tangent at the point $t=x$,

$$\bar{T}(x)=\bar{D}(x)=U\int_0^x v^t\,dt \tag{24}$$

and

$$\bar{T}'(x)=\bar{D}'(x)=v^x U \tag{25}$$

Substituting the value of U from (24) in (25) we find

$$\bar{T}(x)=v^x\,\bar{T}(x)/\int_0^x v^t\,dt$$

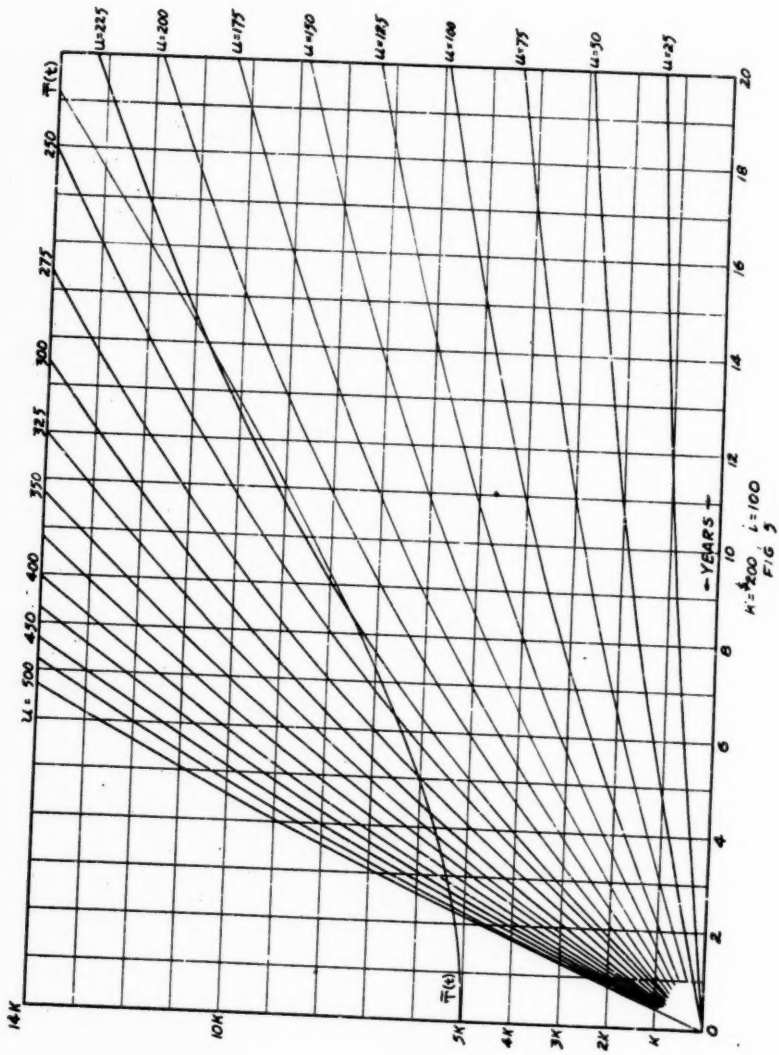
from which

$$\bar{T}(x)\int_0^x v^t\,dt-v^x\,\bar{T}(x)=0.$$

But by (22) this is precisely the condition that U is a minimum.

11. In most cases which arise in practice the analytical method of finding the life-equation of an asset fails owing to the empirical character of the outlay function. The question suggests itself whether a graphic method, similar to that employed in the simpler case treated in 7, can be devised, which will yield an approximate solution of the problem. The theorems of the preceding article offer the key to such a method.

Let us suppose that the total outlay graph has been constructed on a convenient scale, the scale depending on the magnitude of the quantities involved. Every point on this curve determines a definite uniform charge to production curve. We seek that particular one of these curves which is tangent to the total outlay graph. The abscissa of the point of contact would


T(t)
L=225
L=200
L=175
L=150
L=125
L=100
L=75
L=50
L=25
250
275
300
325
350
400
450
500
L=
14K
10K
5K
4K
3K
2K
K
0
2
4
6
8
10
12
14
16
18
20
YEARS
K=200, L=100
FIG 3

give us the life-time of the asset, and the initial slope of the uniform charge to production curve would give us the rate

Instead of constructing first the total outlay curve, we may first prepare a sheet with rulings, as shown in (Fig. 5), each ruling representing the uniform charge to production curve corresponding to a definite U, the successive values U being chosen at suitable intervals. We may then plot the graph of any given outlay function on this same sheet and from among the various rulings select that on which comes nearest having contact with the outlay graph. This will yield a first approximation of x. A closer approximation may then be obtained by the usual processes of interpolation.

Robert E. Moritz

THE SIMULTANEOUS DISTRIBUTION OF MEAN AND STANDARD DEVIATION IN SMALL SAMPLES

By ALLEN T. CRAIG

1. Introduction. If samples of n items are selected at random from a normal universe, it is well known that the arithmetic mean $\bar{x}$ and standard deviation s computed from samples are independent in the probability sense and that the simultaneous frequency distribution is

$$F(\bar{x}, s) = C s^{n-2} e^{-\frac{n s^2 + n\bar{x}^2}{2\sigma^2}}$$

If, however, the parent population is other than the normal type, there appears to be little known regarding the form of $F(\bar{x}, s)$. In the present paper, we propose to determine the simultaneous frequency function of the arithmetic mean and standard deviation in samples of small numbers of items selected at random from a rather arbitrary universe. For convenience, we shall classify frequency distributions according as the range of the independent variable is $(-\infty, \infty)$, $(0, \infty)$ or $(0, a)$, $a > 0$. We shall further assume that the total area under the distribution function is unity.

2. The simultaneous distribution of $\bar{x}$ and s in samples of $n = 2$. Let $f(x)$, $-\infty < x < \infty$ be the frequency function of the variable x. Let x_1 and x_2 be two independently observed values of x. We write

$$x_1 + x_2 = 2\bar{x}$$
$$x_1^2 + x_2^2 = 2s^2 + 2\bar{x}^2$$

We seek the function $F(\bar{x}, s)$ such that $F(\bar{x}, s)d\bar{x}ds$ is, to within infinitesimals of higher order, the probability of the simultaneous occurrence of $\bar{x}$ in $(\bar{x}, \bar{x}+d\bar{x})$ and s in $(s, s+ds)$. For $\bar{x}$ and s assigned, x_1 may have either value $\bar{x}-s$ or $\bar{x}+s$ and x_2 is uniquely determined by $x_2 = 2\bar{x}-x_1$.

$$F(\bar{x}, s)d\bar{x}ds = f(\bar{x}-s)f(2\bar{x}-x_1)dx_1 dx_2$$

Thus

$$+f(\bar{x}+s)f(2\bar{x}-x_1)dx_1 dx_2 .$$

Since $dx_1 dx_2 = 2d\bar{x}ds$ we have

$$F(\bar{x}, s) = 4f(\bar{x}-s)f(\bar{x}+s). \tag{1}$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for $\bar{x}$ assigned, that $s \leq \bar{x}$. Thus (1) is valid for this type of frequency function but the surface is limited by the x-axis and the line $s=\bar{x}$

If $f(x)$ is defined on the interval $(0, a)$, we note, for $\bar{x}$ assigned on $(0, a/2)$, that $s \leq \bar{x}$; and, for $\bar{x}$ assigned on $(a/2, a)$, that $s \leq a-\bar{x}$. Accordingly, for this kind of frequency function, (1) is valid but the surface is limited by the x-axis and the lines $s=\bar{x}$, $s=a-\bar{x}$.

As simple illustrations, let us find the correlation surface for the mean and standard deviation of samples of two items drawn from distributions of various types.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Then

$$F(\bar{x}, S) = \frac{2}{\sigma^2 \pi} e^{-\frac{S^2 + \bar{x}^2}{\sigma^2}},$$

the well known result.

Example 2. Let

$$f(x) = e^{-x}, \qquad 0 \leq x < \infty.$$

Then

$$F(\bar{x}, S) = 4 e^{-2\bar{x}}$$

over the open region of the $\bar{x}s$-plane bounded by the $\bar{x}$-axis and the line $s = \bar{x}$.

Example 3. Let

$$f(x) = \frac{1}{a}, \qquad 0 \leq x \leq a.$$

Then

$$F(\bar{x}, S) = \frac{4}{a^2},$$

over the region of the $\bar{x}s$-plane bounded by the isosceles triangle with sides $s = 0$, $s = \bar{x}$ and $s = a - \bar{x}$. With a uniform distribution proportional to $4/a^2$ over this triangle, it follows incidentally from very elementary geometry that the marginal totals of the distribution of $\bar{x}$ are given by the known values

$$\phi(\bar{x}) = \frac{4}{a^2}\bar{x}, \qquad 0 \leq \bar{x} \leq \frac{a}{2},$$

$$= \frac{4}{a^2}(a - \bar{x}), \qquad \frac{a}{2} \leq \bar{x} \leq a,$$

and that the marginal totals for the distribution of s are given by

$$\psi(s) = \frac{4}{a^2}(a-2s), \qquad 0 \le s \le \frac{a}{2},$$

which is the result given by Rider.[1]

3. The simultaneous distribution of $\bar{x}$ and s in samples of $n=3$. Consider first a frequency function $f(x)$, $-\infty < x < \infty$. We have

$$x_1 + x_2 + x_3 = 3\bar{x},$$

$$x_1^2 + x_2^2 + x_3^2 = 3s^2 + 3\bar{x}^2.$$

Upon eliminating x_3, we have

$$2x_1^2 + 2x_1x_2 + 2x_2^2 - 6\bar{x}x_1 - 6\bar{x}x_2 - 3s^2 + 6\bar{x}^2 = 0.$$

From simple properties of this ellipse, it follows, for assigned $\bar{x}$ and s that x_1 may be chosen arbitrarily from the interval $(\bar{x} - s\sqrt{2}, \bar{x} + s\sqrt{2})$. With x_1 assigned, x_2 must be selected with certainty as either

$$\frac{3\bar{x} - x_1 - \left[6s^2 - 3(x_1 - \bar{x})^2\right]^{\frac{1}{2}}}{2}$$

or

$$\frac{3\bar{x} - x_1 + \left[6s^2 - 3(x_1 - \bar{x})^2\right]^{\frac{1}{2}}}{2}$$

Finally we must have

$$x_3 = 3\bar{x} - x_1 - x_2.$$

[1] P. R. Rider, On the distribution of ratio of mean to standard deviation etc., Biometrika, vol. 21 (1929) pp. 124-141.

Thus

$$F(\bar{x},s)d\bar{x}\,ds = 2\int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} f(x_1)f(x_2)f(x_3)dx_1dx_2dx_3.$$

From

$$x_1 = x_1,$$
$$x_2 = \frac{3\bar{x}-x_1 \pm [6s^2-3(x_1-\bar{x})^2]^{\frac{1}{2}}}{2},$$
$$x_3 = 3\bar{x}-x_1-x_2$$

we obtain

$$dx_1dx_2dx_3 = \frac{9s}{[6s^2-3(x_1-\bar{x})^2]^{\frac{1}{2}}}\,dx_1d\bar{x}\,ds$$
$$= 9s\,dx_1d\bar{x}\,ds/R$$

where

$$R \equiv [6s^2-3(x_1-\bar{x})^2]^{\frac{1}{2}}.$$

Thus

(2)

$$F(\bar{x},s) = 18s\int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} \frac{1}{R} f(x_1) f\left(\frac{3\bar{x}-x_1+R}{2}\right) f\left(\frac{3\bar{x}-x_1-R}{2}\right) dx_1$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for $\bar{x}$ assigned, that $0 \leq s \leq \bar{x}\sqrt{2}$. Thus the surface is limited by the $\bar{x}$ -axis and the line $s = \bar{x}\sqrt{2}$. Moreover, since x_1, x_2, x_3 are non-negative, x_1 may be selected from the interval $(\bar{x} - s\sqrt{2}, x + s\sqrt{2})$ only as long as $s \leq \bar{x}\sqrt{2}/2$. If

$$\bar{x}\ \bar{2}\sqrt{2} \leq s \leq \bar{x}\sqrt{2}.$$

then x_1 may be selected from the intervals

$$\left(0, \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}\right)$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}, \bar{x} + s\sqrt{2}\right).$$

Accordingly, for this type of frequency function,

$$F(\bar{x}, s) = 18s \int_{\bar{x} - s\sqrt{2}}^{\bar{x} + s\sqrt{2}} \frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right) dx_1,$$

$$0 \leq s \leq \frac{\bar{x}\sqrt{2}}{2},$$

(2.1)

$$= 18s \left[\int_0^{\frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x} + s\sqrt{2}} \right]$$

$$\frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right) dx_1,$$

$$\frac{\bar{x}\sqrt{2}}{2} \leq s \leq \bar{x}\sqrt{2}.$$

If $f(x)$ is defined on the interval $(0, a)$, we note:

for $0 \leq \bar{x} \leq a/3$, $\quad 0 \leq s \leq \bar{x}\sqrt{2}$;

for $a/3 \leq \bar{x} \leq 2a/3$, $\quad 0 \leq s \leq \left[2\bar{x}^2 - 2a\bar{x} + \frac{2a^2}{3}\right]^{\frac{1}{2}}$;

for $2a/3 \leq \bar{x} \leq a$, $\quad 0 \leq s \leq (a-\bar{x})\sqrt{2}$.

Thus in this case, the surface is limited by the x-axis, the lines $s = \bar{x}\sqrt{2}$ and $s = (a-\bar{x})\sqrt{2}$ and the hyperbola

$$s = \left[2\bar{x}^2 - 2a\bar{x} + 2a^2/3\right]^{\frac{1}{2}}.$$

(Fig. 1.). Now x, may be selected from the interval $(\bar{x} - s\sqrt{2}, x + s\sqrt{2})$ as long as $s \leq \bar{x}\sqrt{2}/2$ and $s \leq (a-\bar{x})\sqrt{2}/2$. This holds for that part of the surface over the region bounded by OPa. For that part of the surface over the region bounded by OPU, x, may be selected from the intervals

$$\left(0, \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}\right)$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}, \bar{x} + s\sqrt{2}\right).$$

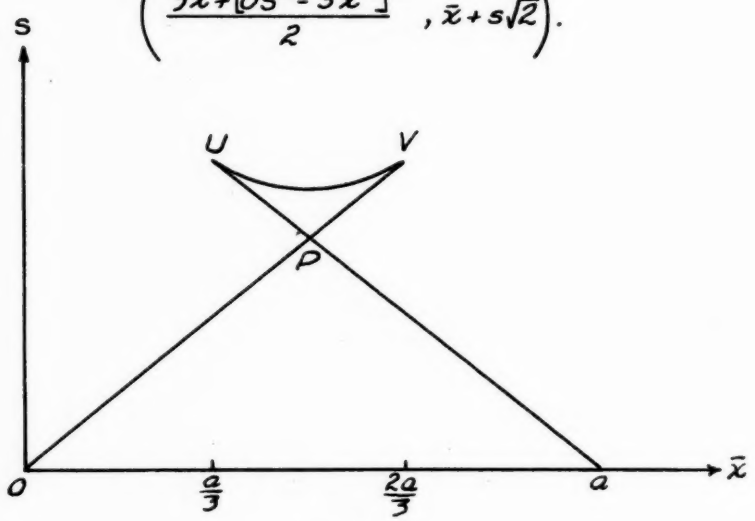


Fig. 1

It is clear that the ranges of arbitrary selection of x_1 for that part of the surface over the region bounded by PVa are

$$\left(\bar{x} - s\sqrt{2},\ \frac{3\bar{x} - a - [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2} \right)$$

and

$$\left(\frac{3\bar{x} - a + [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2},\ a \right).$$

Finally, we find that x_1 may be selected from the intervals

$$\left(0,\ \frac{3\bar{x} - a - [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2}\right),$$

$$\left(\frac{3\bar{x} - a + [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2},\ \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}\right),$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2},\ a\right)$$

for that part of the surface over the region bounded by PUV. If we adopt the notation

$$\phi = \phi(x_1, \bar{x}, s) \equiv \frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right),$$

we have

$$(2.2)\quad F(\bar{x}, s) = 18s \int_{\bar{x} - s\sqrt{2}}^{\bar{x} + s\sqrt{2}} \phi\, dx_1,$$

$$= 18s \left[\int_0^{\frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x} + s\sqrt{2}} \right] \phi\, dx_1,$$

$$= 18s \left[\int_{\bar{x} - s\sqrt{2}}^{\frac{3\bar{x} - a - [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2}} \right.$$

$$\left. + \int_{\frac{3\bar{x} - a + [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2}}^{a} \right] \phi\, dx_1,$$

$$= 18s\left[\int_0^{\frac{3\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}}^{\frac{3\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}}^{a}\right]\phi\, dx_1,$$

for the parts of the surface over the regions indicated above.

In order to illustrate the theory, we shall consider a few examples.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

By (2),

$$F(\bar{x},s) = \frac{3\sqrt{3}}{\sigma^3\sqrt{2\pi}}\, s e^{-\frac{3s^2+3\bar{x}^2}{2\sigma^2}}.$$

Example 2. Let

$$f(x) = e^{-x}, \quad 0 \le x < \infty.$$

By (2.1),

$$F(\bar{x},s) = 6\sqrt{3}\pi s e^{-3\bar{x}}, \qquad 0 \le s \le \frac{\bar{x}\sqrt{2}}{2},$$

$$= 6\sqrt{3}\, s e^{-3\bar{x}}\left[\arcsin\frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \arcsin\frac{\bar{x}}{s\sqrt{2}}\right.$$

$$\left. - \arcsin\frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \frac{\pi}{2}\right], \quad \frac{\bar{x}\sqrt{2}}{2} \le s \le \bar{x}\sqrt{2}.$$

Example 3. Let

$$f(x) = \frac{1}{a}, \qquad 0 \le x \le a.$$

By (2.2),

$$F(\bar{x},s)=\frac{6\sqrt{3}\,\pi s}{a^3}, \text{ over } OPa,$$

$$=\frac{6\sqrt{3}s}{a^3}\left[\arcsin\frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}}+\arcsin\frac{\bar{x}}{s\sqrt{2}}\right.$$

$$\left.-\arcsin\frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}}+\frac{\pi}{2}\right], \text{ over } OPU,$$

$$=\frac{6\sqrt{3}s}{a^3}\left[\arcsin\frac{\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}\right.$$

$$+\arcsin\frac{a-\bar{x}}{s\sqrt{2}}-\arcsin\frac{\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$\left.+\frac{\pi}{2}\right], \text{ over } PVa,$$

$$=\frac{6\sqrt{3}s}{a^3}\left[\arcsin\frac{\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}\right.$$

$$+\arcsin\frac{\bar{x}}{s\sqrt{2}}+\arcsin\frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$+\arcsin\frac{a-\bar{x}}{s\sqrt{2}}$$

$$-\arcsin\frac{\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$\left.-\arcsin\frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}}\right], \text{over } PVU.$$

I have succeeded in obtaining the marginal totals for s from 0 to $a\sqrt{2}/4$ by integrating $F(\bar{x},s)$ with respect to $\bar{x}$ from the boundary (Fig. 1) $s=\bar{x}\sqrt{2}$ to $s=(a-\bar{x})\sqrt{2}$ and obtain as a result the parabola which is known[1] to give the distribution of s from $s=0$ to $s=a\sqrt{6}/6$.

4. The simultaneous distribution of $\bar{x}$ and s in samples of $n=4$. We shall consider first samples of four items drawn from a universe characterized by a law of frequency $f(x)$, $-\infty<x<\infty$. Then

$$x_1+x_2+x_3+x_4=4\bar{x},$$
$$x_1^2+x_2^2+x_3^2+x_4^2=4s^2+4\bar{x}^2.$$

The elimination of x_4 yields

$$x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3-4\bar{x}x_1-4\bar{x}x_2-4\bar{x}x_3-2s^2+6\bar{x}^2=0$$

It follows from the properties of this ellipsoid that x_1 may be chosen arbitrarily from the interval $(\bar{x}-s\sqrt{3},\ \bar{x}+s\sqrt{3})$. For x_1 assigned, the region of arbitrary selection of x_2 is determined by the properties of the ellipse and is

$$\left(\frac{4\bar{x}-x_1-2\left[6s^2-2(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{3},\ \frac{4\bar{x}-x_1+2\left[6s^2-2(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{3}.\right)$$

Upon solving for x_3 in terms of x_1 and x_2 we have

$$x_3=\frac{4\bar{x}-x_1-x_2\pm\left[8s^2-8\bar{x}^2+8\bar{x}x_1+8\bar{x}x_2-2x_1x_2-3x_1^2-3x_2^2\right]^{\frac{1}{2}}}{2}$$

while x_4 is uniquely determined by $x_4=4\bar{x}-x_1-x_2-x_3$.
If we write

$$T\equiv\left[8s^2-8x^2+8\bar{x}x_1+8\bar{x}x_2-2x_1x_2-3x_1^2-3x_2^2\right]^{\frac{1}{2}}$$

and

$$\phi\equiv f(x_1)f(x_2)f\left(\frac{4\bar{x}-x_1-x_2+T}{2}\right)f\left(\frac{4\bar{x}-x_1-x_2-T}{2}\right)$$

[1] H. L. Rietz [Paper to appear presently in Biometrika].

then

$$(3)\qquad F(\bar{x},s)=32s\int_{\bar{x}-s\sqrt{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\frac{1}{7}\Phi\,dx_2\,dx_1.$$

The integration can be carried out in an obvious manner when $f(x)$ is the normal frequency function.

In case $f(x)$ is defined on the interval $(0,\infty)$, we note, for $\bar{x}$ assigned, that $s\le\bar{x}\sqrt{3}$. Thus the surface is limited by the $\bar{x}$-axis and the line $s=\bar{x}\sqrt{3}$ Moreover, x_1 may be selected from the interval $(\bar{x}-s\sqrt{3},\ \bar{x}+s\sqrt{3})$ with x_2 chosen as above only as long as $s\le\bar{x}\sqrt{3}/3$ If $\bar{x}\sqrt{3}/3\le s\le\bar{x}$, then x_1 may be chosen from either of the two intervals

$$\left(0,\ \frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}\right)$$

and

$$\left(\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3},\ \bar{x}+s\sqrt{3}\right)$$

with x_2 chosen as above; or x_1 may be selected from the interval

$$\left(\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3},\ \frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}\right)$$

with x_2 taken from either

$$\left(0,\ \frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}\right)$$

or

$$\left(\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2},\ \frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}\right).$$

when $\bar{x}\le s\le\bar{x}\sqrt{3}$ we may have

$$0\le x_1\le 2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}$$

and

$$2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}} \leq x_1 \leq \frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}$$

with either

$$0 \leq x_2 \leq \frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}$$

or

$$\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2} \leq x_2 \leq \frac{4\bar{x}-x_1+2[6s-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}$$

Or we may have

$$\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3} \leq x_1 \leq \bar{x}+s\sqrt{3}$$

with

$$\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3} \leq x_2 \leq \frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}$$

Accordingly, for this kind of frequency function,

$$F(\bar{x},s)=32s\int_{\bar{x}-s\sqrt{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-x)^2]^{\frac{1}{2}}}{3}} \frac{1}{T}\phi\,dx_2\,dx_1, \tag{3.1}$$

$$0 \leq s \leq \frac{\bar{x}\sqrt{3}}{3},$$

$$=32s\left[\int_0^{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\right.$$

$$\left.+\int_{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\right.$$

$$+\int_{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}$$

$$+\int_{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\Bigg]\frac{1}{T}\phi\,dx_2\,dx_1,$$

$$\frac{\bar{x}\sqrt{3}}{3}\le s\le\bar{x},$$

$$=32s\Bigg[\int_{0}^{2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}}\int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}$$

$$+\int_{0}^{2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}}\int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$+\int_{2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}$$

$$+\int_{2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$+\int_{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\Bigg]\frac{1}{T}\phi\,dx_2\,dx_1,$$

$$\bar{x}\le s\le\bar{x}\sqrt{3}$$

By similar reasoning, the writer has determined $F(\bar{x}, s)$ for $n = 4$ and $f(x)$ defined on the interval $(0, a)$. The results, however, are quite lengthy and formal and will not be presented here.

Allen T. Craig.

THE LIMITS OF A MEASURE OF SKEWNESS

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The measure of skewness

$$\underline{S} = \frac{\text{mean -- median}}{\text{standard deviation}}$$

is sometimes recommended because of its simplicity. Obviously neither this nor any other statistic can be of much value until something at least is known of its distribution in samples from populations of some plausible form. For populations near the normal form the inefficiency of the median as a statistic of location suggests that the standard error of $\underline{S}$ may be considerably greater than that of μ_3/σ^3. We know of no investigation of the sampling distribution of $\underline{S}$. Apparently even the range is unknown. The object of the present note is to show that $\underline{S}$ necessarily lies between -1 and 1.

The proof consists of three successive transformations of the sample, each increasing $\underline{S}$, which nevertheless in the end remains less than unity.

1. Without loss of generality let us suppose that the median is zero and that the mean $\bar{x}$ is positive. Taking

$$\sigma^2 = \frac{\Sigma(\underline{x} - \bar{\underline{X}})^2}{\underline{n}} = \frac{\Sigma x^2}{n} - \bar{x}^2,$$

n being the number of observations, which we suppose odd, we have

$$\underline{S} = \bar{x}/\sigma.$$

If a negative observation $-\underline{a}$ be replaced by zero, the mean is increased by $\underline{a}/n$. In the second of the expressions above for σ^2 the mean of the squares is diminished by $\underline{a}^2/n$, while on account of the change in the mean, a further subtraction is made. Thus σ diminishes. Hence $\underline{S}$ increases if we alter the distribu-